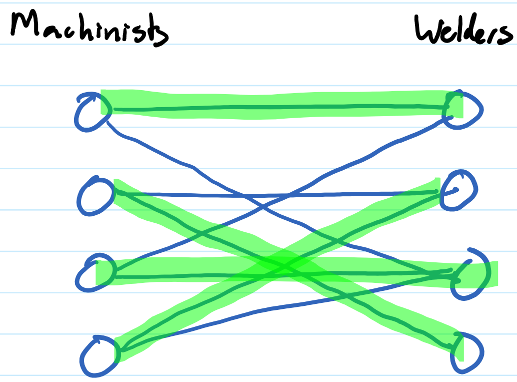
Bipartite Matching

$X \cup Y = V$
$X \cap Y = \emptyset$

Bipartite Graph: V can be partitioned into X, Y s.t. all edges (u,v) have $u \in X$, $v \in Y$ or $u \in Y$, $v \in X$

Matching in a graph:



there is a perfect matching for this graph

Problem: Find a matching of maximum size

Maximum Flow

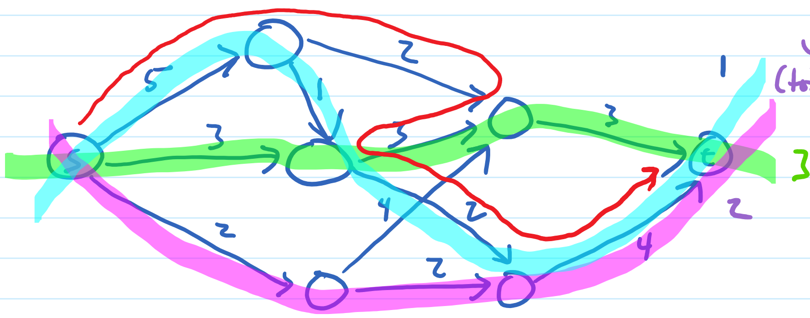
Problem: Given directed graph G with source s (no edges in) and sink t (no edges out) and capacity $c(e) > 0$ for each $e \in E$,

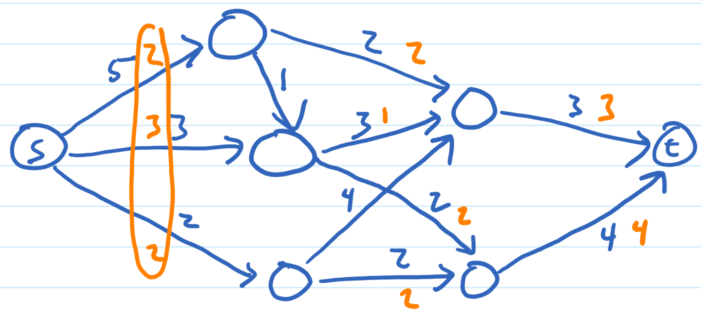
find flow of maximum value

flow: assignment of $f(u,v) \geq 0$ to each edge satisfying $f(u,v) \leq c(u,v)$ for each $(u,v) \in E$ (capacity)

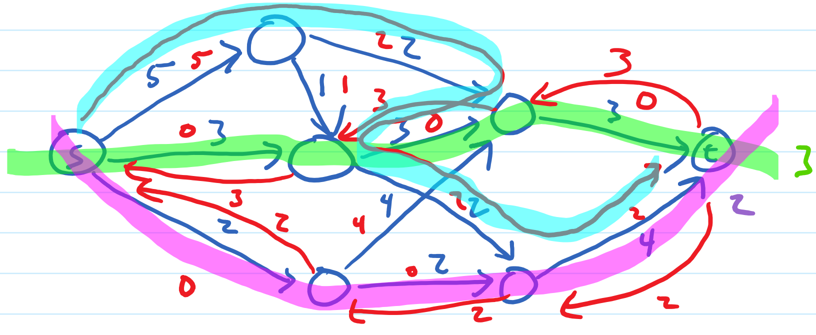
and for all $v \in V - \{s,t\}$ $\underbrace{\sum_{(u,v)\in E} f(u,v)}_{f^{in}(v)} = \underbrace{\sum_{(v,x)\in E} f(v,x)}_{f^{out}(v)}$ (conservation)

$v(f) = f^{out}(s)$ (total flow)

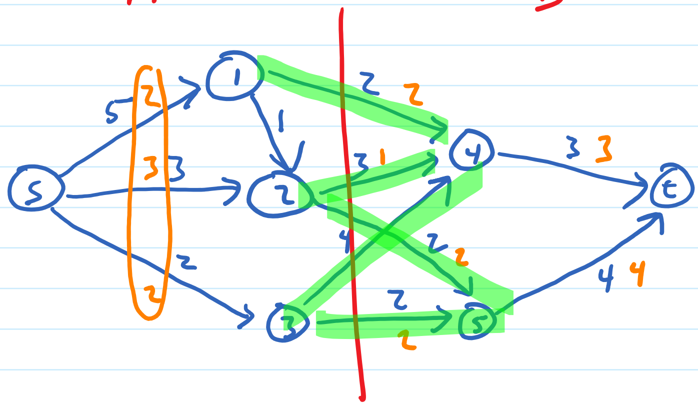




$v(f) = 7$



A B

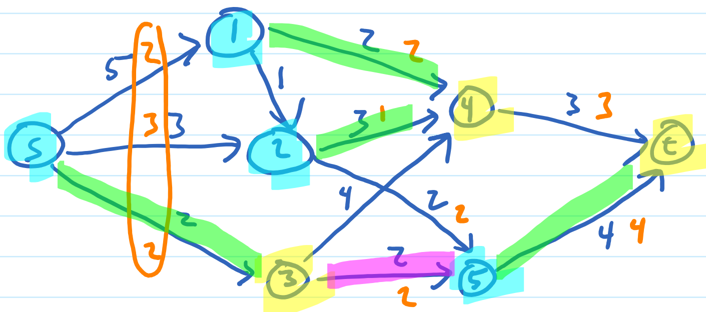


$v(f) = 7$

$f^{out}(A) = 7$ $f^{in}(B) = 7$

$f^{in}(A) = 0$ $f^{out}(B) = 0$

$f^{out}(A) - f^{in}(A) = 7 - 0 = 7 = v(f)$



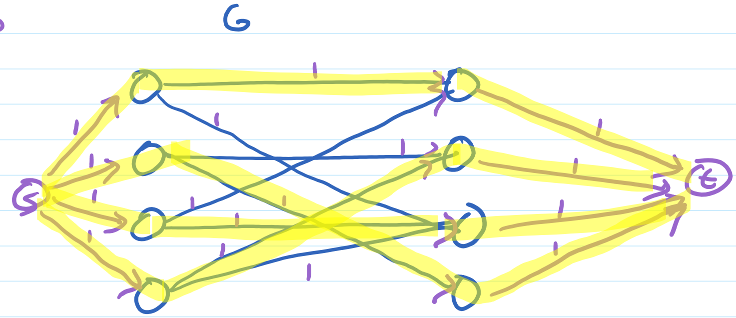
A $f^{out}(A) = 9$

B $f^{in}(A) = 2$

$f^{out}(A) - f^{in}(A) = 9 - 2 = 7 = v(f)$

Bipartite Matching solved by Maximum Flow

G′: input to max flow
needs source
sink
capacities
directions



MAX-BIPARTITE-MATCH (G)

construct G′ as above
find max flow f for G′
return $M = \{ (u,v) \mid u \in X, v \in Y, f(u,v) = 1 \}$

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$
$\Updownarrow$
There is a matching M in G with $|M| = k$

Proof: a little later

LEMMA 2: For directed graph G with integer capacities, there is a max flow w/ int-valued flows

Proof: more later

THM: For bipartite G, max flow f in G′ gives max matching

Proof: Let f be max flow, M be corresponding matching

f is integer-valued L2

$|M| = v(f)$ L1

Suppose M not maximum: M' is matching with $|M'| > |M|$

Then there is corresponding flow f' with
$v(f') = |M'| > |M| = v(f)$

So f is not max flow $\Rightarrow\Leftarrow$ f is max flow

So M is maximum

DEF: s-t cut is a partition of V into A, B s.t. $s \in A$, $t \in B$

DEF: $f^{in}(X) = \sum_{\substack{(u,v)\in E \\ u \notin X \\ v \in X}} f(u,v)$ $f^{out}(X) = \sum_{\substack{(u,v)\in E \\ u \in X \\ v \notin X}} f(u,v)$

THM: Let f be a flow, (A,B) be an s-t cut. Then $f^{out}(A) - f^{in}(A) = v(f)$

Proof: $v(f) = f^{out}(s)$ def

THM. Let f be a flow, (A,B) be an s-t cut. Then $f^{out}(A) - f^{in}(A) = v(f)$

Proof:

$$v(f) = f^{out}(s) \qquad \text{def}$$
$$= f^{out}(s) - f^{in}(s) \qquad \text{no edges to } s \text{ so } f^{in}(s) = 0$$

$$f^{out}(v) - f^{in}(v) = 0 \text{ for all } v \in A - \{s\} \qquad \text{conservation (and } t \notin A)$$

$$v(f) = \sum_{v \in A} f^{out}(v) - f^{in}(v) \qquad v \neq s \text{ terms are zero}$$

$$= \sum_{v \in A} \sum_{(v,x) \in E} f(v,x) - \sum_{(u,v) \in E} f(u,v) \qquad \text{def } f^{out}, f^{in}$$

$$= \sum_{v \in A} \sum_{\substack{(v,x) \in E \\ x \notin A}} f(v,x) - \sum_{\substack{(u,v) \in E \\ u \notin A}} f(u,v) \qquad \text{edges } A \to A \text{ cancel}$$

$$= f^{out}(A) - f^{in}(A) \qquad \text{rearrange; def}$$

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$\Updownarrow$

There is a matching M in G with $|M| = k$

Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}$

M is a matching in G

$(x,y) \in M \rightarrow (x,y) \in G$ — construction of G' (no edges $X \to Y$ added)

can't have $(x,y_1), (x,y_2) \in M, y_1 \neq y_2$ — $f(s,x) = f^{in}(x) = f^{out}(x) \geq f(x,y_1) + f(x,y_2) = 1+1=2$

can't have $(x_1,y), (x_2,y) \in M, x_1 \neq x_2$

conservation, def f^{out}, def of M

$\Rightarrow\Leftarrow$ since $c(s,x) = 1$

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$$v(f) = f^{out}(A) - f^{in}(A) = f^{out}(A) \qquad \text{prev THM, construction of } G'$$

$$= \sum_{\substack{(x,y) \in E' \\ x \in A \\ y \notin A}} f(x,y) \qquad \text{def}$$

$$= \sum_{\substack{(x,y) \in E' \\ x \in X \\ y \in Y \\ f(x,y) = 1}} f(x,y)$$

$$= |\{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}|$$

$$= |M|$$

$\Leftarrow$ similar but backwards