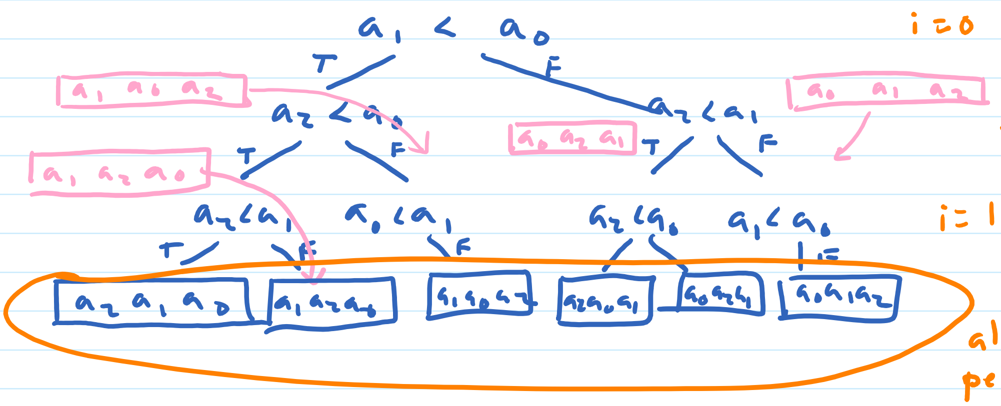
Lower Bounds

Lower bound for problem P is f(n) means
no alg that solves P has worst case better than f(n)

Lower bound for sorting

```
for i = 0 to n-2
  for j = 0 to n-2-i
    if (a[j+1] < a[j]) then
      swap(a, j, j+1)
```

Decision Tree



$i = 0 \quad j = 0$

$i = 0 \quad j = 1$

$i = 1 \quad j = 0$

all $3! = 6$ permutations of a_0, a_1, a_2

DT for sorting n items using comparisons must have $\geq n!$ leaves

Binary tree with $n!$ leaves must have height $\geq \log_2 n!$

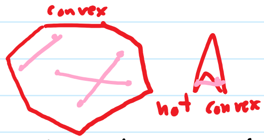
So any comparison-based sort must have a worst case $\geq \log_2 n!$

$\log_2 n + \log_2 n-1 + \dots + \log_2 1$

$O(n \log n)$

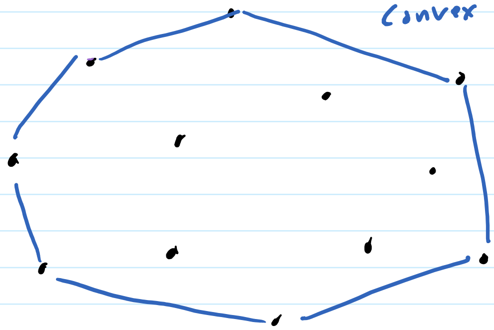
Reductions

Convex Hull



line between interior pts doesn't leave

Given a set of points in the plane, what convex polygon with vertices chosen from the set contains the rest?



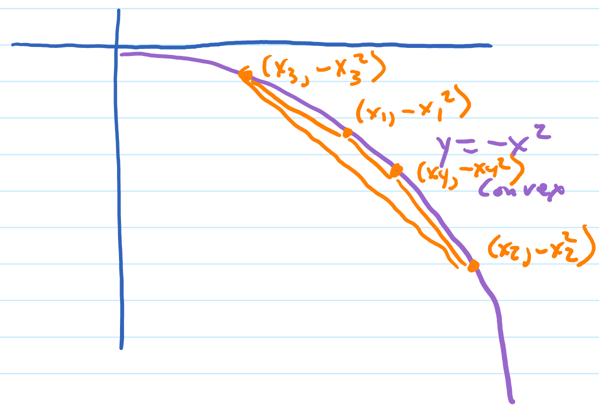
Sorting w/ Convex Hull

```
CH-SORT( A )

    for each x ∈ A
        pts ← pts ∪ { (x, -x^2) }

    ch = CONVEX-HULL(pts)

    sort ← [ ]
    for each (x,y) ∈ ch
        sort ← sort + [x]
        fix wraparound
    return sort
```



Lower bound on CH: $O(n \log n)$ (losing comparisons)

if CH is easy then sorting is easy
(w.c. better than $O(n \log n)$)

if sorting is hard then convex hull is hard

HAMILTONIAN-CYCLE $\le$ LONGEST-PATH

Solving HC using LP:

<u>HC(G)</u>

k ← # verts in G

long ← LONGEST-PATH(G)

if

NP: a class of useful problems

NP-complete: the hardest of the problems in NP

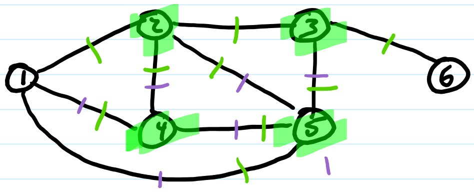
For decision problems A and B, $A \le_P B$ means

Vertex Cover and Independent Set

subset of V
s.t. each edge has
at least one endpoint in the subset

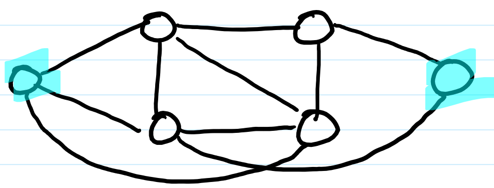
Vertex Cover: Given G and k, determine if there is a vertex cover of size $\le k$

k=2? NO
k=3? NO
k=4?



subset of vertices
with no edges
between members of subset

Independent Set: Given G and k, determine if there is an independent set of size $\ge k$



Algorithm for VC: return IS(G, n−k)

$VC \le_p IS$ (if VC is hard then IS is hard)

THM: G has VC of size k if and only if G has IS of size n − k

Proof: $\Rightarrow$ Suppose G has VC C of size k. [need to show there is IS of size n−k]

Let $S = C^c = V - C$. Certainly size of $S = n-k$
[need also that S is an IS]
for all $u,v \in S$
(u,v) is not an edge

Let $u, v \in S$ (need (u,v) not an edge)

Suppose $(u,v) \in E$
C covers (u, v)
$u \in C$ or $v \in C$
so $u \notin S$ or $v \notin S$ $\Rightarrow\Leftarrow$

$\therefore (u,v) \notin E$

So for all $u,v \in S$, $(u,v) \notin E$, so S is an independent set

$\Leftarrow$ Later

$A \le_p B$: there is an alg for A that runs in polynomial # of steps including calls to alg for B (that count as a single step)

$O(n^k)$ for some k ← there is a alg for B w/ polynomial worst-case

THM: If $A \leq_p B$ and $B \in P$ then $A \in P$

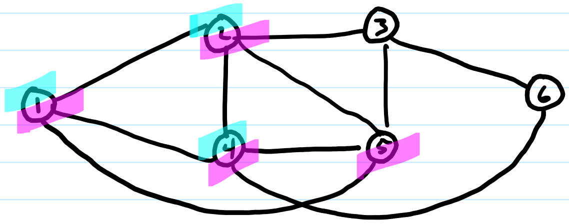
COR: If $A \leq_p B$ and $A \notin P$ then $B \notin P$

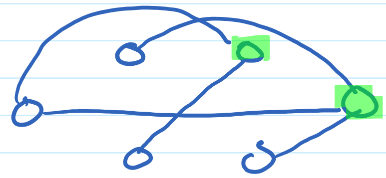
Vertex Cover and Clique

subset of verts w/ all possible edges between

Clique: Given G and k, determine if G contains a clique of size $\geq k$.

$k=3$
$k=4$





edges exactly where G doesn't have them

Alg for VC: return CLIQUE(G^c, $n-k$)

poly-time poly-time poly-time

VC $\leq_P$ CLIQUE

THM: G has VC of size k if and only if G^c has clique of size $|V|-k$

Proof: $\Rightarrow$ Suppose G has VC C of size k. Let $C' = V - C$. So size of $C' = n-k$

[need that C' is a clique
for all $u,v \in C'$ (u,v) is in G^c]

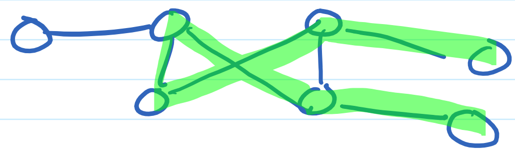
Let $u, v \in C'$ and $u \neq v$.
- Suppose (u,v) is not an edge in G^c
- Then (u,v) is an edge in G — def of G^c
- So $u \in C$ or $v \in C$ — C covers edges of G
- So $u \notin C'$ or $v \notin C'$ $\Rightarrow\Leftarrow$ — construction of C'

$\therefore$ (u,v) is an edge in G^c, so for all $u,v \in C'$, edge exists, so C' is a clique

$\Leftarrow$ Suppose G^c has clique C of size k. Let $C' = V - C$.

Let (u,v) be an edge in G.
- Suppose $u \notin C'$ and $v \notin C'$
- Then $u \in C$ and $v \in C$
- So (u,v) is an edge in G^c.
- So (u,v) is not an edge in G

Hamiltonian Path and Long Path and Travelling Salesperson



no HP

Cycle / cycle

Hamiltonian Path: Given G, is there a simple path through all vertices?

Long Path: Given G, k, is there a simple path of length $\geq k$?

$HP \leq_p LP$: $\underline{HP(G)}$: return $LP(G, n)$

$HP \leq_p HC$?

every possible edge exists

cycle that visits each vert once

Travelling Salesperson: Given complete weighted G, and bound k, is there a tour of total weight $\leq k$?

$HC \leq_p TSP$:

1) Create G' w/ same vertex set as G and all possible edge — poly

2) set $\ell(u,v) = \begin{cases} 0 & \text{if } (u,v) \in G \\ 1 & \text{otherwise} \end{cases}$ — poly

3) return $TSP(G', 0)$