3-SAT and Vertex Cover

SAT : Given Boolean formula φ, does it have a satisfying assignment?

assignment of T/F to vars that makes φ true

$x_1 \wedge \sim x_1$ $\qquad ((x_1 \vee x_2) \rightarrow (x_1 \wedge x_3)) \wedge \sim x_1 \wedge \sim x_3$

F $\vee$ F

F $\quad$ T $\quad$ T $\quad$ T

$x_1 = F \quad x_2 = F \quad x_3 = F$

3-SAT : Given Boolean formula φ in 3-CNF form, is it satisfiable?

φ is $C_1 \wedge C_2 \wedge \cdots \wedge C_n$

each clause is $t_1 \vee t_2 \vee t_3$ where $t_i = x_k$ or $\sim x_k$

$(\sim x_1 \vee \sim x_2 \vee \sim x_3) \wedge (\sim x_1 \vee \sim x_2 \vee x_3) \wedge (\sim x_1 \vee x_2 \vee \sim x_3) \wedge (\sim x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \sim x_2 \vee \sim x_3) \wedge (x_1 \vee x_2 \vee \sim x_3) \wedge (x_1 \vee x_2 \vee x_3)$

$(x_1 \vee x_2 \vee x_2)$

in poly-time

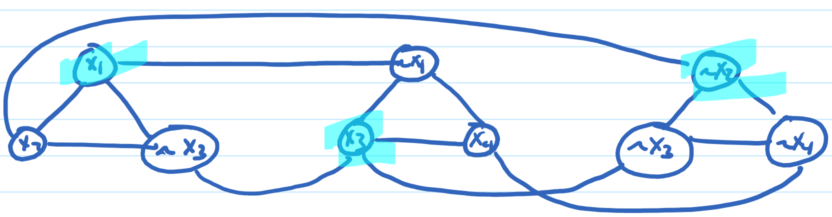
3-SAT $\leq_p$ IND-SET [Goal: given φ in 3-CNF, construct G and pick k s.t. φ is satisfiable $\longleftrightarrow$ G has a IS of size $\geq k$]

1) construct G, pick k
2) return IS(G, k)

T $\quad$ T $\quad$ T

$(x_1 \vee x_2 \vee \sim x_3) \wedge (\sim x_1 \vee x_3 \vee x_4) \wedge (\sim x_2 \vee \sim x_3 \vee \sim x_4)$

$x_1 = T$
$x_2 = F$
$x_3 = T$



P vs NP

P: set of problems w/ poly-time solutions

NP: set of problems w/ poly-time verification algorithms

Ex: TSP (Given G, k, determine if tour of cost $\le k$)

HVN PVG JAC NUQ FRA GRU BOG HVN $k = 28000$

HVN FRA PVG JAC NUQ BOG GRU HVN

Prob A is in NP means there is a poly-time verification alg A' s.t. set of inputs x to A for which A(x) = YES is exactly the set ~~set~~ of 1st inputs to A' for there is poly-size y 2nd input s.t. A'(x, y) = OK

$\underbrace{\text{TSP-VERIFY}(G,k}_{\text{original input}},\ t)$: (t = certificate, evidence)

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compute cost c of tour t
if c ≤ k return OK
else     return NOT-OK
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VC-VERIFY(G, k, C):

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for each edge (u,v) in G
    if u ∉ C and v ∉ C
        return NOT-OK
return OK
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3-SAT-VERIFY(φ, A):

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for each clause C
    for each term t in C
        if A makes t T goto X
    return NOT-OK
  X:
return OK
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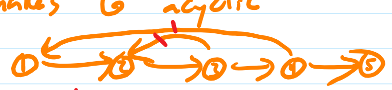
IND-SET: certificate/evidence is the independent set

HAM-PATH: evidence is the Hamiltonian path

MIN-FEEDBACK-ARC-SET: evidence is set of edges

given directed G, int k is there a set of $\le k$ edges that when removed makes G acyclic

$k = 2$



YES

NP-complete: hardest problems in NP

DEF: A is NP-complete means 1) $A \in NP$
2) for all $B \in NP$, $B \le_p A$

THM: If $X \in NP$ and Y is NP-complete and $Y \le_p X$ then X is also NP-complete

Proof: 1) $X \in NP$

2) Let $B \in NP$ [need $B \le_p A$]

Then $B \le_p Y$ — Y is NP-complete; part 2 def NP-complete

So $B \le_p X$ — transitivity

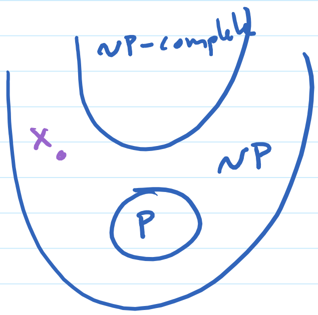
3-SAT ⇢ HP ⇢ HC ⟶ TSP

CIRCUIT-SAT ⇢ 3-SAT ⟶ IS ⟶ VC ⟶ CLIQUE

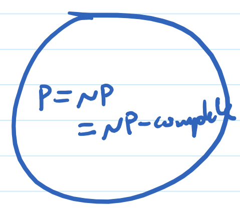
THM: $P \subseteq NP$

Let $X \in P$. X-VERIFY(x, y): if $\overbrace{X(x)}^{\text{poly-time alg for X}} =$ YES return OK

else return NOT-OK



or



To show $P \ne NP$, it is sufficient to find super-poly lb for some $X \in NP$

To show $P = NP$, it is sufficient to find poly-time for some NP-complete X