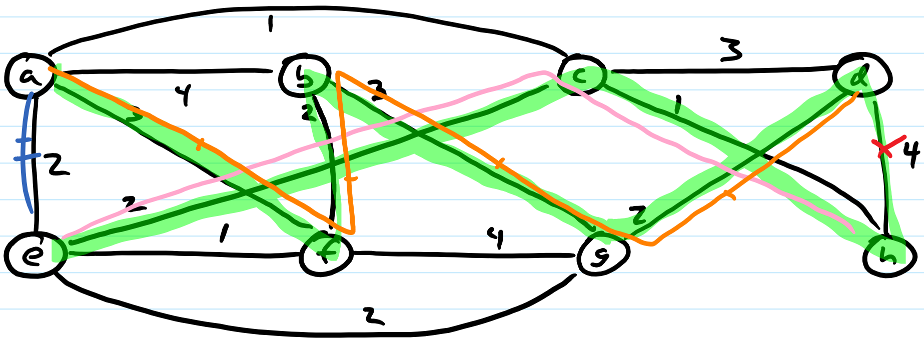
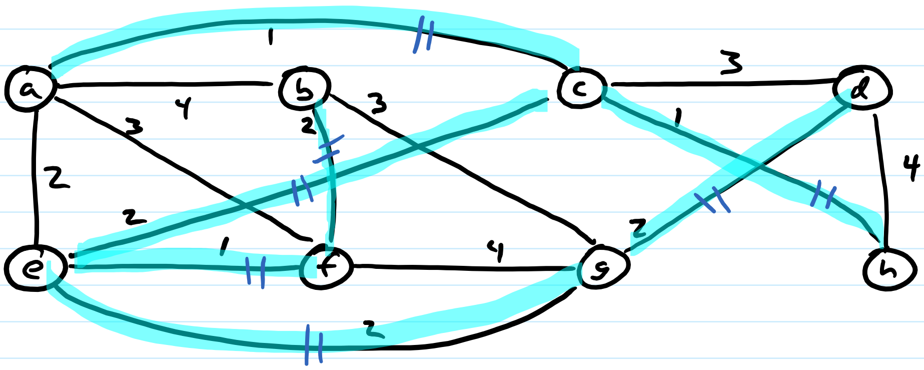
Minimum Spanning Tree : MST of undirected, connected, weighted graph G is a set of edges T ⊆ E that forms a tree (acyclic), connects all vertices in V, and minimizes total weight



spanning tree
weight = ~~17~~ 15

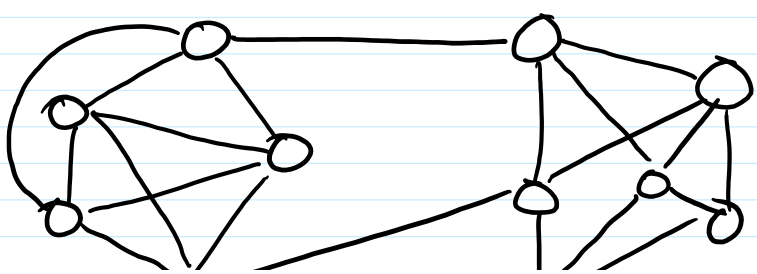


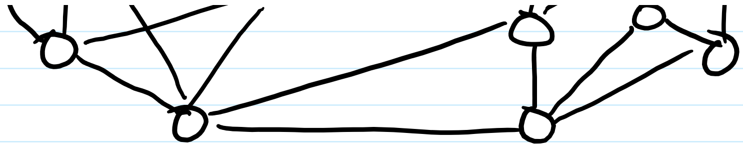
minimum spanning tree
weight = 11

Cut : Partition of vertices into two sets
↳ collection of disjoint, non-empty subsets s.t. their union is entire set

Cut Property : If all edge weights are distinct and S, V−S is a cut then the minimum weight edge across S, V−S is in every MST
↳ one endpoint in S, one in V−S

S V − S



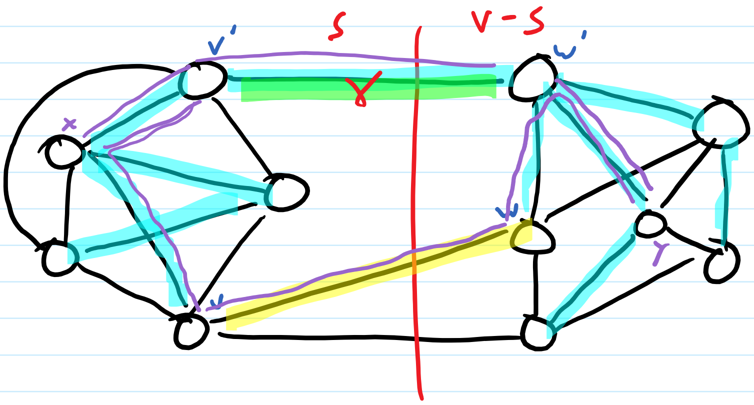


Proof: Let $G=(V,E)$ be a graph with distinct <u>non-negative</u> weights $S, V-S$ be a cut, and $e=(v,w)$ be the min-weight edge across that cut.

[want: $\forall$ MSTs T, $(v,w)\in T$
$\forall\ T$, if T is MST then $(v,w)\in T$
$\forall\ T$, if $(v,w)\notin T$ then T not MST]

Suppose T is a spanning tree of G and T does not contain (v,w).

[want to find spanning tree T' s.t. $w(T') < w(T)$]



v, w are connected in T via P — T is a spanning tree

let (v', w') be 1^{st} edge along path P $v \rightsquigarrow w$ from S to $V-S$

let $T' = T - \{(v',w')\} \cup \{(v,w)\}$

T' spans G

let $x, y \in V$
x, y are connected in T via path P' $x, u_1, \dots, u_k, y$

2 cases) 1) no edge in P' is (v', w')
so all still in T', so x,y still connected

2) P' $x, u_1, \dots, v', \cancel{w'}, \dots, u_k, y$
$v \rightarrow w$

$x \rightsquigarrow y$ in T' via stitched-together path

T' has same number of edges as T
\# verts in $T = n-1$
\# verts in $T' = n-1-1+1 = n-1$

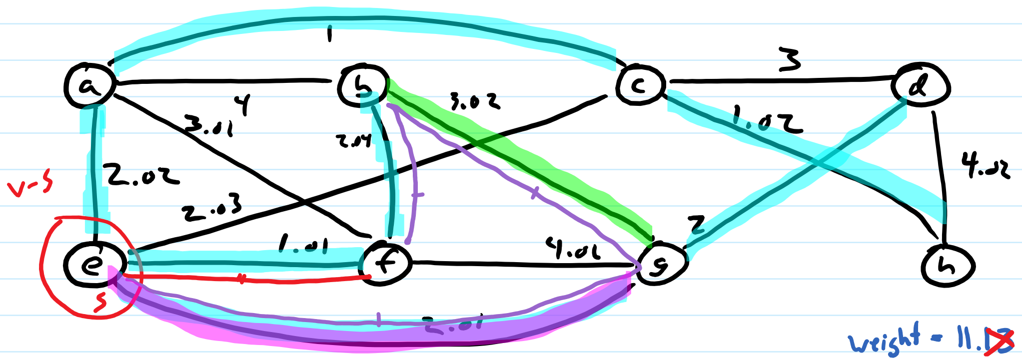
T' is acyclic
(spanning and $n-1$ edges $\rightarrow$ tree $\rightarrow$ acyclic) [202]

T' is a spanning tree

$w(T') < w(T)$

$\therefore$ T is not a minimum spanning tree

$w(T') = w(T) + w(v,w) - w(v',w')$
$< w(T)$ since $w(v,w) < w(v',w')$
since both cross $S, V-S$
and (v,w) is min. weight such edge



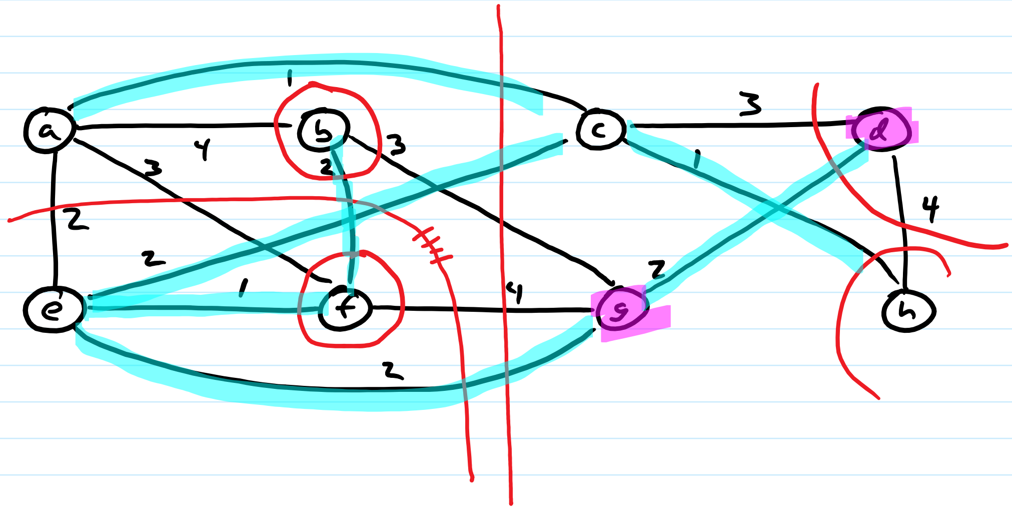
— spanning tree that does not include (e,f) which is min-weight across S: $\{e\}$, $V-S$: $\{a,b,c,d,f,g,h\}$

the path $e \rightsquigarrow f$ in that spanning tree

(e,g) is 1st edge in that path across cut

swap (e,g) for (e,f) to find a better spanning tree - original was not MST

Light Edge Theorem



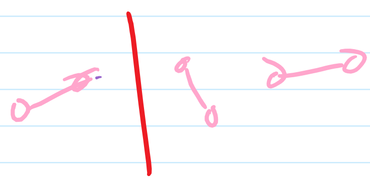
cut respects tree T : $(u,v) \in T \longrightarrow (u \in S \leftrightarrow v \in S)$

Proto-MST: a subset A of edges s.t. $A \subseteq T$ for some MST A.

subset of a MST

Light Edge Theorem: Let A be a proto-MST, $(S, V-S)$ be a cut that A respects, and (u,v) be min weight edge across $(S, V-S)$. Then $A' = A \cup \{(u,v)\}$ is a proto-MST

Proof: similar to cut property



GENERIC-MST

```
T ← ∅
while |T| < |V| - 1
    S,V-S ← some cut that respects T
    (u,v) ← light edge S → V-S
    T ← T ∪ {(u,v)}
```

Kruskal's Algorithm

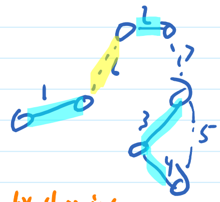
n = # vertices
m = # edges

O(m log n)

Kruskal's Algorithm : consider edges in order of ↑ weight
add edge if connects two different components of proto-MST

subset of some MST

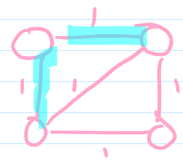
DFS? O(n+m)
for every edge, so total
O(n·m+m²) need to do better

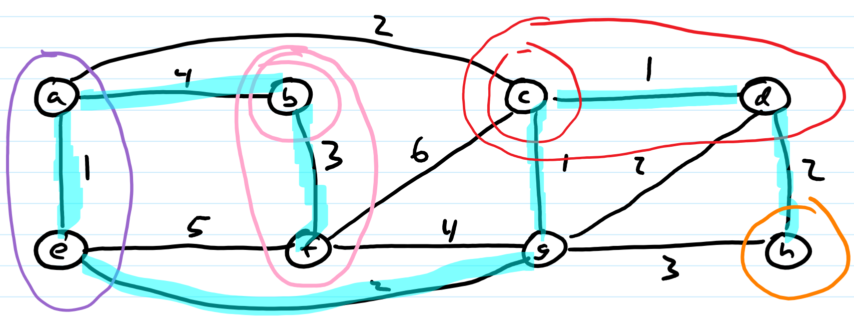


INV: a) for all edges (x,y) before current in ordered list, x⇝y using selected edges

b) selected edges form proto-MST

show this part is maintained by showing selected edge (v, w) light weight edge across cut comp(v), rest of graph





(a,e) (c,d) (c,g) (d,g) (a,c) (e,g) (d,h) (b,f) (g,h) (a,b) (f,g) (c,f) (e,f)