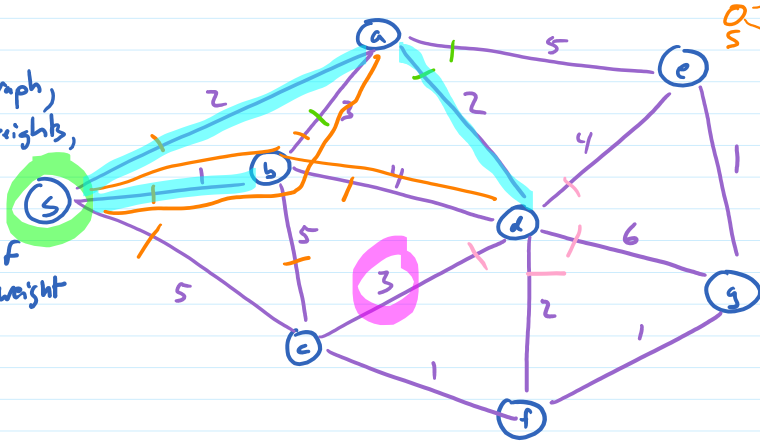
Shortest Paths

given weighted graph, with non-neg weights, and 2 vertices v, ~~w~~

find v→w of minimum total weight "shortest"



a	2/s
b	1/s
c	5/s
d	~~5/b~~ 4/a
e	7/a
f	6/d
g	10/d

shortest path of all :	s	0	s
next shortest :	sb	1	b
	sa	2	a
	sad	4	d
	sc	5	c
	scf	6	f
	sae		e

Dijkstra(G, ℓ)

$S \leftarrow \{s\}$

$d(s) \leftarrow 0$

while $S \neq V$

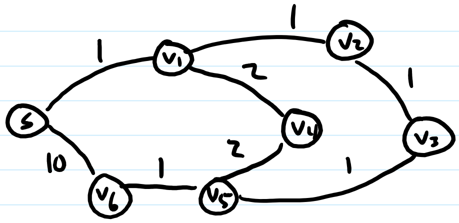
choose $v \notin S$ to minimize $d'(v) = \min_{\substack{u \in S \\ (u,v) \in E}} d(u) + \ell(u,v)$

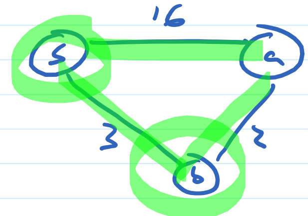
Shortest Paths

no neg-weight edges

Given weighted G (directed or undirected), and a source vertex s, find min-weight path $s \rightsquigarrow v$ for all vertices v.

Consider no path to have distance ∞





cost of $s \rightsquigarrow u \to v$
=
cost of shortest path $s \rightsquigarrow u$
+ cost of $u \to v$

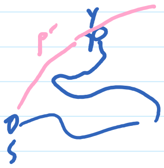
Dijkstra(G, ℓ)
$S \leftarrow \{s\}$
$d(s) \leftarrow 0$
while $S \neq V$
choose $v \notin S$ to minimize $d'(v) = \min_{\substack{u \in S \\ (u,v) \in E}} d(u) + \ell(u,v)$

$S \leftarrow S \cup \{v\}$
$d(v) = d'(v)$

$Q \leftarrow \emptyset$ — priority queue key = ~~tot~~ weight of shortest path so far from $s \rightsquigarrow v$
$S \leftarrow \{s\}$ — verts we know shortest path to
cost of shortest paths: $d(s) \leftarrow 0$
$\pi(s) \leftarrow NIL$ — $\pi[v]$ = next to last vertex on shortest $s \rightsquigarrow v$
for $v \in V, v \neq S$
    if $(s,v) \in E$
        $d'[v] \leftarrow \ell(s,v)$ — d' = copies of priorities
        $\pi[v] \leftarrow v$
    else
        $d'[v] \leftarrow \infty$
        $\pi[v] \leftarrow NIL$
    Q.enqueue$(v, d'[v])$
n iterations: while $Q \neq \emptyset$
    $v = Q$.extractMin()
    $[v] \leftarrow d'[v]$
    $S \leftarrow S \cup \{v\}$
m total iterations: for $(v,w) \in E$ where $w \in Q$
        if $d[v] + \ell(v,w) < d'(w)$
            $d'[w] = d[v] + \ell(v,w)$
            $\pi[w] = v$
            Q.decreasePriority$(w, d'[w])$

total = n · extractMin time + m · changePriority time

INVARIANT:
a) $s \in S$
b) S, Q partition V
c) $|Q| = |V| - 1 - \#$ iterations
d) for $v \in S$, $d[v] = \delta(s,v)$ — tot weight of shortest $s \rightsquigarrow v$ — d, π is correct
$\pi[v]$ = next-to-last on min-cost path
e) for $v \in Q$, $d'[v]$ = cost of min-cost $s \rightsquigarrow v$ using int. verts in S
$\pi[v]$ = next-to-last on that path
$d'[v]$ = priority of v
f) $d'[v]$ is the priority of v in Q
g) $\pi[v] = NIL$ or $\pi[v] \in S$ for all $v \in V$



p'
y
R
O
S