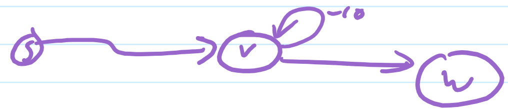
Shortest Paths

Pre: no neg-weight cycles (all shortest paths algs)
: no neg-weight edges (Dijkstra's)



Dijkstra (G, ℓ)

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Q ← ∅                      priority queue   key = ~~tot~~ weight of shortest path so far from s ⇝ v
S ← {s}                    verts we know shortest path to
d[s] ← 0
π[s] ← NIL                 π[v] = next to last vertex on shortest s ⇝ v
  v ∈ V, v ≠ S
  if (s,v) ∈ E
        d'[v] ← ℓ(s,v)     d' = copies of priorities
        π[v] ← v              = tot weight of shortest path seen so far
  else
        d'[v] ← ∞
        π[v] ← NIL
  Q.enqueue(v, d'[v])
→ while Q ≠ ∅
    v = Q.extractMin()
    d[v] ← d'[v]           d[v] = weight of shortest path s ⇝ v
    S ← S ∪ {v}
    for (v,w) ∈ E where w ∈ Q
        if d[v] + ℓ(v,w) < d'(w)
            d'[w] = d[v] + ℓ(v,w)
            π[w] = v
            Q.decreasePriority(w, d'[w])
```

~~a~~ 0/-

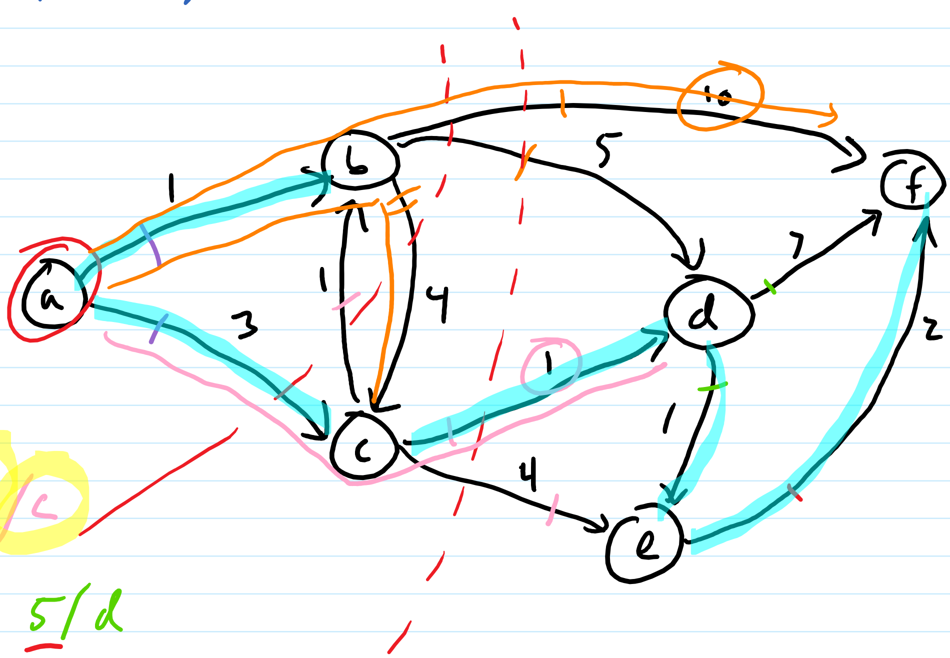
~~b~~ ~~∞/-~~ 1/a

~~c~~ ~~∞/-~~ 3/a

~~d~~ ~~∞/-~~ ~~6/b~~ 4/c

~~e~~ ~~∞/-~~ ~~7/c~~ 5/d

f ~~∞/-~~ ~~11/b~~ 7/e



Invariant

$Q \leftarrow \emptyset$
$S \leftarrow \{s\}$
$d[s] \leftarrow 0$
$\pi[s] \leftarrow NIL$
  $v \in V, v \neq s$
  if $(s,v) \in E$
    $d'[v] \leftarrow \ell(s,v)$
    $\pi[v] \leftarrow v$
  else
    $d'[v] \leftarrow \infty$
    $\pi[v] \leftarrow NIL$
  $Q.enqueue(v, d'[v])$
while $Q \neq \emptyset$
  $v = Q.extractMin()$
  $d[v] \leftarrow d'[v]$
  $S \leftarrow S \cup \{v\}$
  for $(v,w) \in E$ where $w \in Q$
    if $d[v] + \ell(v,w) < d'(w)$
      $d'[w] = d[v] + \ell(v,w)$
      $\pi[w] = v$
      $Q.decreasePriority(w, d'[w])$

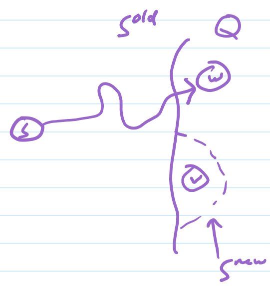
INVARIANT:
a) $s \in S$
b) S, Q partition V
c) $|Q| = |V| - 1 - \#$ iterations
d) for $v \in S$, $d[v] = \delta(s,v)$ ← tot cost of shortest $s \rightsquigarrow v$
   $\pi[v]$ = next-to-last on min-cost path
   d, π correct
e) for $v \in Q$, $d'[v]$ = cost of min-cost $s \rightsquigarrow v$ using int. verts in S
   $\pi[v]$ = next-to-last on that path
   $d'[v]$ = priority of v
   d', π are correct
f) $d'[v]$ is the priority of v in Q
g) $\pi[v] = NIL$ or $\pi[v] \in S$ for all $v \in V$

e) $d'[w]$ are tot weight of shortest path $s \xrightarrow{\text{all verts in } S} w$
  1) $d'[v]$ is tot weight of a path $s \xrightarrow{S} w$
  2) $d'[v] \leq$ tot weight of any path $s \xrightarrow{S} w$

Let P be any path $s \xrightarrow{S^{new}} w$
  3 cases

i) P doesn't use v
  then P is $s \xrightarrow{S^{old}} w$

$d'_{new}[w] \leq d'_{old}[w] \leq \ell(P)$
  ↑ code ↑ inv

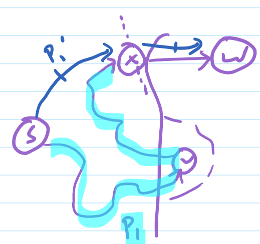


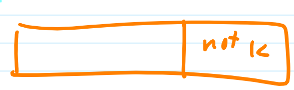
ii) P is $s \rightsquigarrow v \to w$
  let $\underline{P'}$ be shortest path $s \rightsquigarrow v$
  + edge $v \to w$

$\ell(P) = \ell(P_1) + \ell(v,w)$
$\geq \ell'(P_1) + \ell(v,w)$
$= d[v] + \ell(v,w)$
$\geq d'_{new}[w]$

iii) P is $s \xrightarrow{S_{old}} v \xrightarrow{S_{old}} w$
  Let P' be shortest path $s \rightsquigarrow x$
  + edge $x \to w$

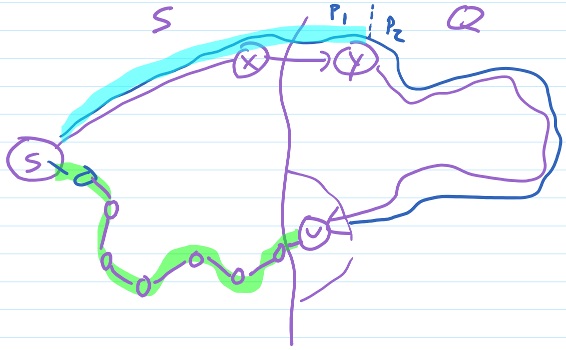
$\ell(P) = \ell(P_1) + \ell(x,w)$
$\geq \ell(P_1') + \ell(x,w)$
$= d[x] + \ell(x,w)$
$\geq d'_{old}[w]$
$\geq d'_{new}[w]$





inv: count = len(last run)
no runs of len(k)
before current run

d) $d[v]$ is lot weight shortest path $s \to v$ (for all $v \in S$)



$d_{new}[v] = d'[v]$

$d'[v] \le d'[y]$ extract-min

$d'[y] \le \ell(P_1)$ INV e

$\ell(P_1) \le \ell(P_1) + \ell(P_2)$ PRE no neg-weight edges so $\ell(P_2) \ge 0$

$= \ell(P)$

Negative Weights

Negative weights $\longrightarrow$ Dijkstra