Bipartite Matching

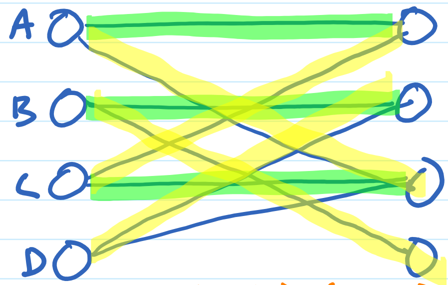
$X \cap Y = \emptyset$ $\quad X, Y \neq \emptyset$
$X \cup Y = V$

Bipartite Graph: V can be partitioned into X, Y s.t. all edges (u,v) have $u \in X$ and $v \in Y$ or $u \in Y$ and $v \in X$

Matching in a bipartite graph: subset of edges M s.t.

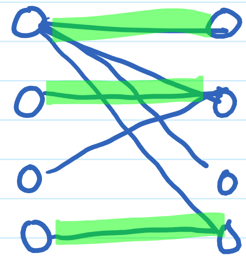
$(x_1, y_1) \in M$
$(x_2, y_2) \in M$
$\downarrow$
$x_1 \neq x_2$ and
$y_1 \neq y_2$
(or $x_1 = x_2$ and $y_1 = y_2$)

X Machinists — Y Welders



(A, Y) (C, W) (B, Z) (D, X)

edge (m, w) means m will work with w and vice versa



Problem: find matching of maximum size

Maximum Flow

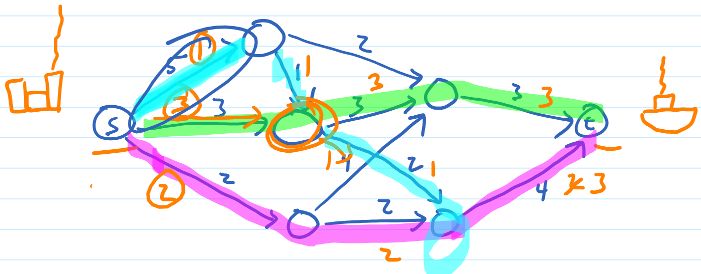
Problem: Given directed graph G with source s (no edges in) and sink t (no edges out) and capacity $c(e) > 0$ for each $e \in E$, find flow of maximum value

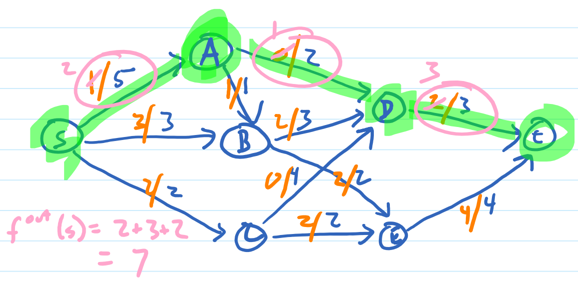
flow: assignment of $f(u,v) \geq 0$ to each edge

such that $f(u,v) \leq c(u,v)$ for each $(u,v) \in E$ (capacity)

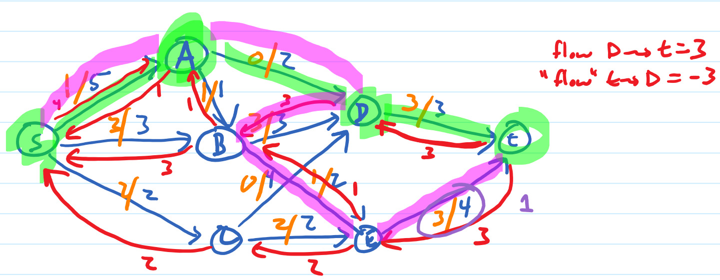
and for all $v \in V - \{s,t\}$, $\underbrace{\sum_{(u,v)\in E} f(u,v)}_{f^{in}(v)} = \underbrace{\sum_{(v,u)\in E} f(v,u)}_{f^{out}(v)}$ (conservation)

value of flow $v(f) = \sum_{(s,x)} f(s,x) = f^{out}(s)$



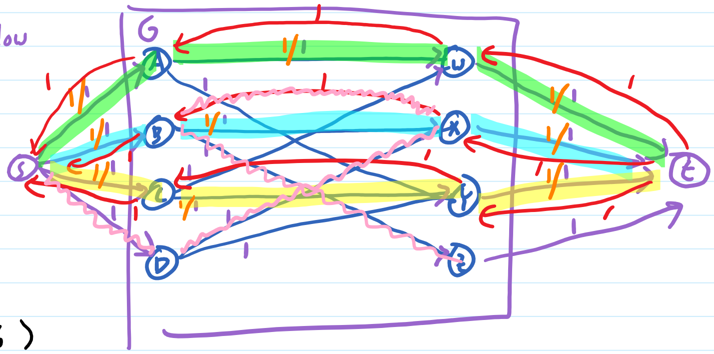


residual capacity



Bipartite Matching solved by Maximum Flow

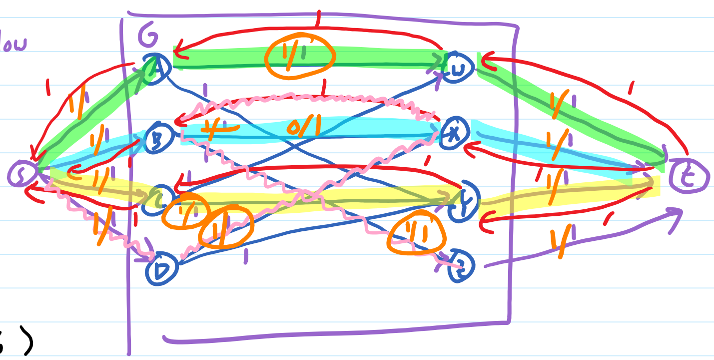
G' : input to max flow
needs
source
sink
capacity
direction



MAX-BIPARTITE-MATCH (G)

1) construct G' as above
2) find max flow f of G'
3) output $M = \{ (x,y) \in G \text{ s.t. } f(x,y) = 1 \}$

low

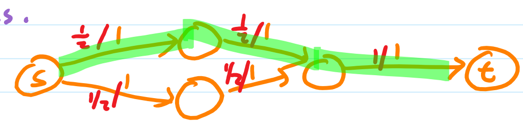


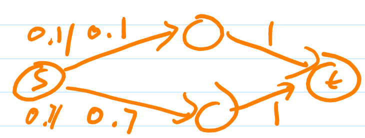
$M = \{ (A,U), (C,Y), (B,Z), (D,X) \}$

G)

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$
$\Updownarrow$
There is a matching M in G with $|M| = k$

LEMMA 2: For directed graph G with integer capacities, then there is a max flow that has integer capacities.





THM: For bipartite G, max flow f in G' gives max matching
Proof: Let f be max flow, M be corresponding matching

f is integer-valued — L2

$|M| = v(f)$ — L1 ↓

Suppose M not maximum: M' is matching with $|M'| > |M|$

Then there is corresponding flow f' with $v(f') = |M'| > |M| = v(f)$ — L1 ↑

So f is not max flow $\Rightarrow\Leftarrow$

∴ So M is maximum

DEF: s-t cut is

THM: Let f be a flow, (A,B) be an s-t cut.

Proof:

$$v(f) = f^{out}(s)$$
$$= f^{out}(s) - f^{in}(s)$$

$f^{out}(v) - f^{in}(v) = 0$ for all $v \in A - \{s\}$

$$v(f) = \sum_{v \in A} f^{out}(v) - f^{in}(v)$$

$$= \sum_{v \in A} \sum_{(v,x) \in E} f(v,x) - \sum_{(u,v) \in E} f(u,v)$$

$$= \sum_{v \in A} \sum_{\substack{(v,x) \in E \\ x \notin A}} f(v,x) - \sum_{\substack{(u,v) \in E \\ u \notin A}} f(u,v)$$

$$= f^{out}(A) - f^{in}(A)$$

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$$\Updownarrow$$

There is a matching M in G with $|M| = k$

Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}$

M is a matching in G

$(x,y) \in M \rightarrow (x,y) \in G$

can't have $(x,y_1), (x,y_2) \in M$, $y_1 \neq y_2$

can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$$v(f) = f^{out}(A)$$

$$= \sum_{\substack{(x,y) \in E' \\ x \in A \\ y \notin A}} f(x,y)$$

$$= \sum_{\substack{(x,y) \in E' \\ x \in X \\ y \in Y \\ f(x,y) = 1}} f(x,y)$$

$$= |\{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}|$$

$$= |M|$$