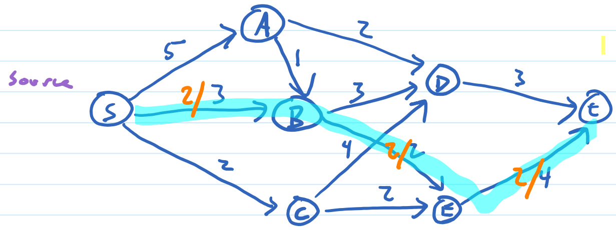
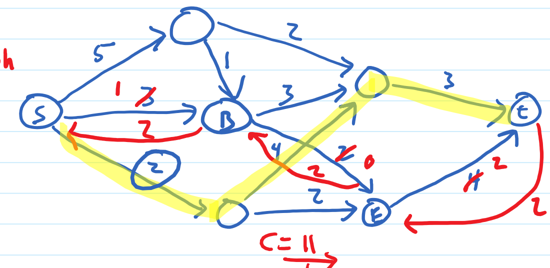
Ford-Fulkerson

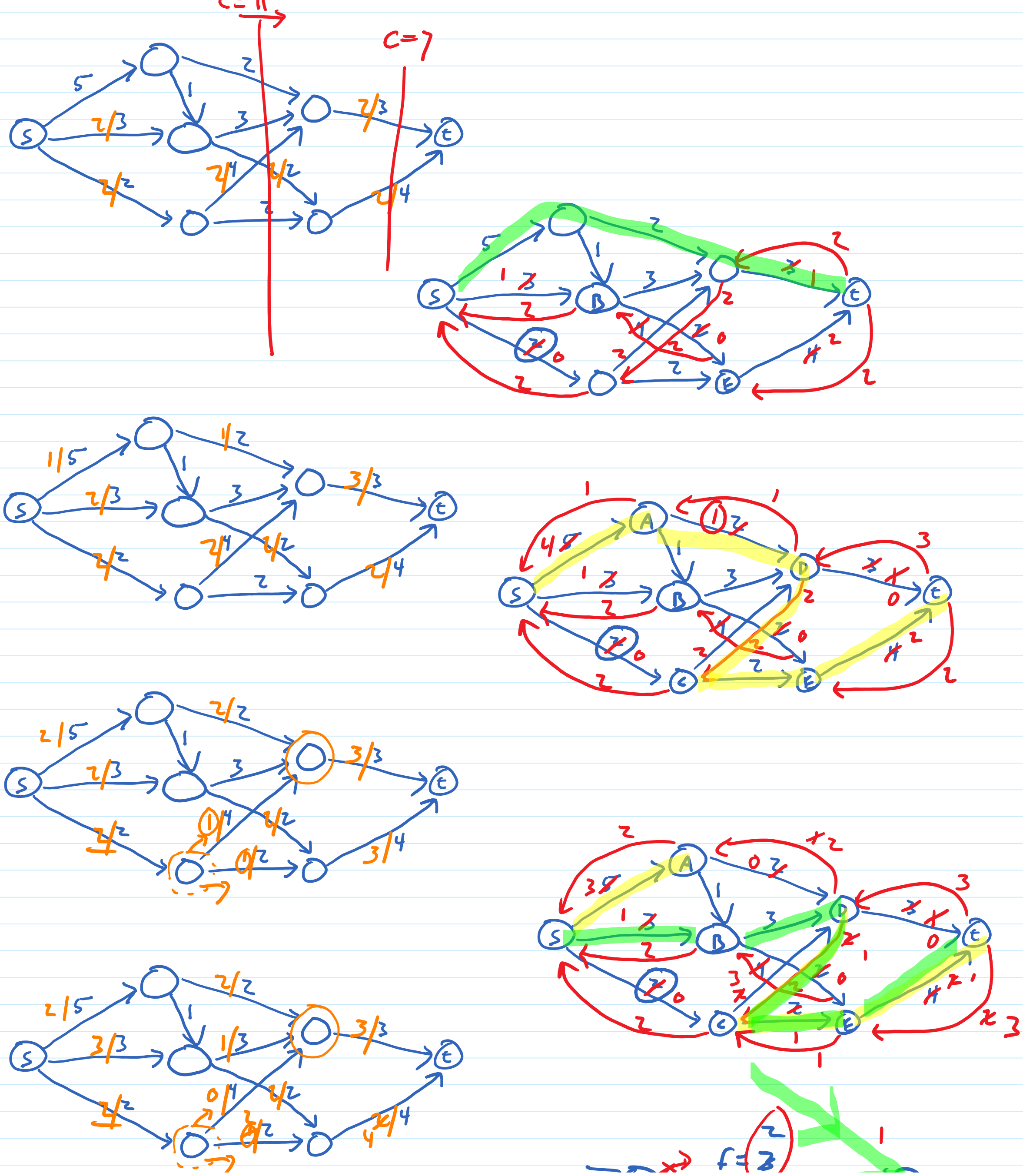


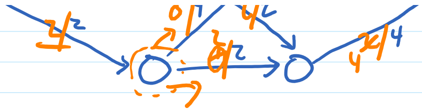
add flow of 2 along this path

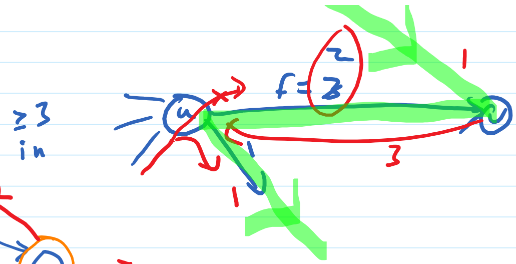
residual graph

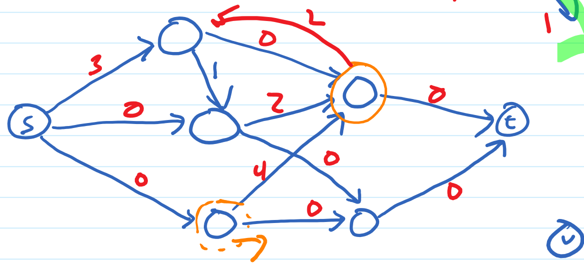


add flow of 2 along this path









MAX-FLOW-FF(G)

PRE: edge in ≤1 dir between u, v
capacities are pos. integer
s is source
t is sink
and all v on some path s ⇝ t

$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with backward edges w/ $c_r(u,v)=0$
while there is a path P s ⇝ t in G_r with all edges $(u,v) \in P$ s.t. $c_r(u,v)>0$
$O(m)$ find such a path ← BFS $O(n+m) = O(m)$
C = # iterations; b ← bottleneck(P) $O(n)$ (min $c_r(u,v)$ over edges $(u,v) \in P$)
C ≤ v(f) — min of ints is an int
for $(x,y) \in P$ — $O(n)$
if (x,y) is forward (is in G) — $O(1)$
$f(x,y) \leftarrow \underbrace{f(x,y)}_{\text{int}} + \underbrace{b}_{\text{int}}$
else
$f(y,x) \leftarrow f(y,x) - b$
$c_r(x,y) \leftarrow c_r(x,y) - b$
$c_r(y,x) \leftarrow c_r(y,x) + b$
return f

$O(n)$

total $O(C \cdot m)$ pseudopolynomial

exponential in # bits in weights

INVARIANT: 1) f is flow
2) G_r is residual graph for G, f ← a) $c_r(u,v) = c(u,v) - f(u,v)$
$c_r(v,u) = f(u,v)$
for all (u,v) in G
3) f, c_r all integer-valued

Basis: 1) 0 is flow, 2) $c_r(u,v) = c(u,v)$
$= c(u,v) - f(u,v)$ since $f(u,v)=0$
$c_r(v,u) = 0$
3) $0 \in \mathbb{Z}$ $= f(u,v)$ since $f(u,v)=0$

Maintenance: 3)

2) for (x,y) modified in loop, if (x,y) is forward $f_{new}(x,y)$

$c_{r\,new}(x,y) =$

=

=

$c_{r\,new}(y,x) =$

if (x,y) backward

1) capacity : if (u,v) appears forward in P

$\rightarrow \left[f_{new}(u,v) \le c(u,v) \right]$

where $f_{new}(u,v) = f_{old}(u,v) + b$

$\boxed{0 < b \le c_r(u,v)} = c(u,v) - f_{old}(u,v)$

choice of b (min c_r over edges in P)

$f_{new}(u,v) = f_{old}(u,v) + b \le c(u,v) - \cancel{f_{old}(u,v)} + \cancel{f_{old}(u,v)}$

AND $f_{old}(u,v) \ge 0$ and $b \ge 0$

so $\underbrace{f_{old}(u,v) + b}_{f_{new}(u,v)} \ge 0$

if (u,v) appears backwards in P

$0 \le b \le c_r(v,u) =$

conservation : 4 cases a) enter, leave v along forward

b) enter v along backward, leave v along forward

c)

d)

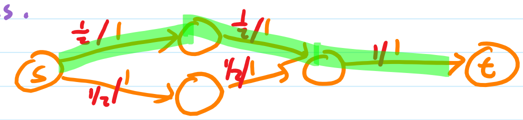
Bipartite Matching solved by Maximum Flow

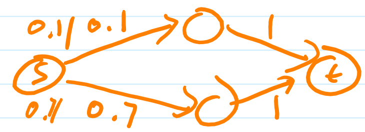
LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$\Updownarrow$

There is a matching M in G with $|M| = k$

LEMMA 2: For directed graph G with integer capacities, then there is a max flow that has integer capacities.





THM: For bipartite G, max flow f in G' gives max matching

Proof: Let f be max flow, M be corresponding matching

f is integer-valued — L2

$|M| = v(f)$ — L1 $\downarrow$

Suppose M not maximum: M' is matching with $|M'| > |M|$

Then there is corresponding flow f' with — L1 $\uparrow$

$v(f') = |M'| > |M| = v(f)$

So f is not max flow $\Rightarrow\Leftarrow$

$\therefore$ So M is maximum

DEF: s-t cut is partition of V into A, B s.t. $s \in A$, $t \in B$

THM: Let f be a flow, (A,B) be an s-t cut. Then $f^{out}(A) - f^{in}(A) = v(f)$

$\qquad f^{in}(B) - f^{out}(B)$

Proof:

$$v(f) = f^{out}(s) \qquad \text{def}$$
$$= f^{out}(s) - f^{in}(s) \qquad \text{no edges in so } f^{in}(s) = 0$$

$$f^{out}(v) - f^{in}(v) = 0 \text{ for all } v \in A - \{s\} \qquad \text{conservation}$$

$$v(f) = \sum_{v \in A} f^{out}(v) - f^{in}(v) \qquad v \neq s \text{ terms are } 0$$

$$= \sum_{v \in A} \sum_{(v,x) \in E} f(v,x) - \sum_{(u,v) \in E} f(u,v) \qquad \text{def } f^{out}, f^{in}$$

$$= \sum_{v \in A} \sum_{\substack{(v,x) \in E \\ x \notin A}} f(v,x) - \sum_{\substack{(u,v) \in E \\ u \notin A}} f(u,v) \qquad \text{edges } A \to A \text{ cancel}$$

$$= f^{out}(A) - f^{in}(A) \qquad \text{rearrange; def } f^{in}(A), f^{out}(A)$$

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$\Updownarrow$

There is a matching M in G with $|M| = k$

Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}$

M is a matching in G

$(x,y) \in M \rightarrow (x,y) \in G$ — no edges added between X, Y

can't have $(x,y_1), (x,y_2) \in M$, $y_1 \neq y_2$ — otherwise $f(x,y_1) = f(x,y_2) = 1$, so $f^{out}(x) \geq 2$ so $f^{in}(x) \geq 2$ so $f(s,x) = 2$ $\Rightarrow\Leftarrow$

can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$ — similar

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$v(f) = f^{out}(A) - f^{in}(A) = f^{out}(A)$ — prev THM; const. of G' allows no edges into A

$= \sum_{\substack{(x,y) \in E' \\ x \in A \\ y \notin A}} f(x,y)$ — def

$= \sum_{\substack{(x,y) \in E' \\ x \in X \\ y \in Y \\ f(x,y)=1}} f(x,y)$ — all other flows are 0

$= |\{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}|$

$= |M|$ — substitution

$\Leftarrow$ similar

Minimum Cut

THM: If f is a flow s.t. there is no path $s \rightsquigarrow t$ in corresponding G_r, then there is a s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

$$c(A^*, B^*) = \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} c(u,v)$$

Proof: Later

COR: MAX-FLOW-FF returns a maximum flow

Proof: Let f be the flow returned by MAX-FLOW-FF

There is no path $s \rightsquigarrow t$ in corresponding G_r — stopping condition in code

Find s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$ — THM

Let f' be any flow.

$$\begin{aligned}
v(f') &= f'^{out}(A^*) - f'^{in}(A^*) && \text{THM} \\
&\le f'^{out}(A^*) && f'^{in}(A^*) \ge 0 \\
&= \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} f'(u,v) && \text{def } f^{in} \\
&\le \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} c(u,v) && \text{capacity} \\
&= c(A^*, B^*) = v(f) && \text{def } c(A^*,B^*)\text{; choice of } A^*, B^*
\end{aligned}$$

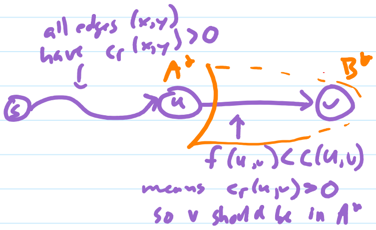
THM: If f is a flow s.t. there is no path $s \rightsquigarrow t$ in corresponding G_r, then there is a s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

Proof: Construct $A^* = \{ v \mid \exists \text{ path } s \rightsquigarrow v \text{ in } G_r \}$
$B^* = V - A^*$

(A^*, B^*) in an s-t cut — $s \rightsquigarrow s$; $s \not\rightsquigarrow t$ means $t \notin A^*$ so $t \in B^*$

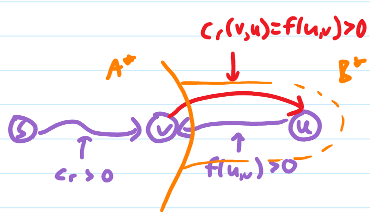
Consider $(u,v) \in E$ where $u \in A^*$, $v \in B^*$

Suppose $f(u,v) < c(u,v)$
then $c_r(u,v) = c(u,v) - f(u,v) > 0$
so there is a path $s \rightsquigarrow v$ using edges having $c_r > 0$
$\therefore v \in A^*$ $\Rightarrow\Leftarrow$ (contradicts $v \in B^*$)

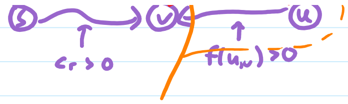


Consider $(u,v) \in E$ where $u \in B^*$, $v \in A^*$

Suppose $f(u,v) > 0$
then $c_r(v,u) = f(u,v) > 0$
so there is a path $s \rightsquigarrow u$ using edges having $c_r > 0$



then $c_f(v,u) = f(u,v) > 0$
so there is a path $s \rightsquigarrow u$ using edges
having $c_f > 0$
$\therefore\ u \in A^*$ $\Rightarrow\Leftarrow$ (contradicts $u \in B^*$)



$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$ conservation

$$= \sum_{\substack{(u,v)\in E\\ u\in A^*\\ v\in B^*}} f(u,v) - \sum_{\substack{(u,v)\in E\\ u\in B^*\\ v\in A^*}} f(u,v)$$ def f^{in}, f^{out}

$$= \sum_{\substack{(u,v)\in E\\ u\in A^*\\ v\in B^*}} c(u,v) - 0$$ ✱ above

$$= c(A^*, B^*)$$ def capacity across cut

constructed as in prev thm

COR: Let f be max flow in G and (A^*, B^*) be corresponding cut. Then (A^*, B^*) is a min-capacity cut

Proof: