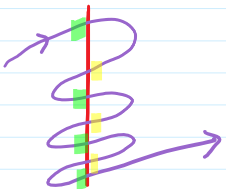
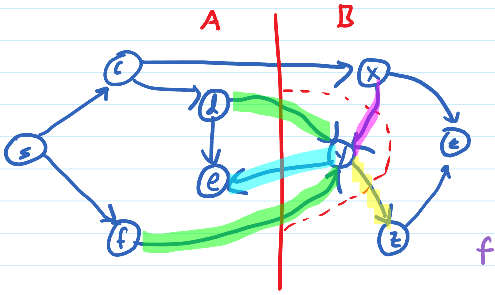
Value of flow = Flow across cut

DEF: s-t cut is partition of V into A, B s.t. $s \in A$, $t \in B$

THM: Let f be a flow, (A,B) be an s-t cut. Then $\underline{f^{out}(A)} - \underline{f^{in}(A)} = \underline{v(f)}$





$$v(f) = f^{out}(A) - f^{in}(A)$$

$$f^{out}(A') = \underline{f^{out}(A) - f(A,y) + f(y,B)}$$

$$f^{in}(A') = \underline{f^{in}(A) - f(y,A) + f(B,y)}$$

$$f^{out}(A') - f^{in}(A') =$$

$$\underbrace{f^{out}(A) - f^{in}(A)}_{v(f)} - \underbrace{f(A,y) - f(B,y)}_{f^{in}(y)} + \underbrace{f(y,B) + f(y,A)}_{f^{out}(y)}$$

$$v(f) - \underbrace{f^{in}(y) \qquad + f^{out}(y)}_{}$$

$$v(f) - 0 = v(f)$$

Proof:

$$v(f) = f^{out}(s)$$ def $v(f)$

$$= f^{out}(s) - f^{in}(s)$$ $f^{in}(s) = 0$

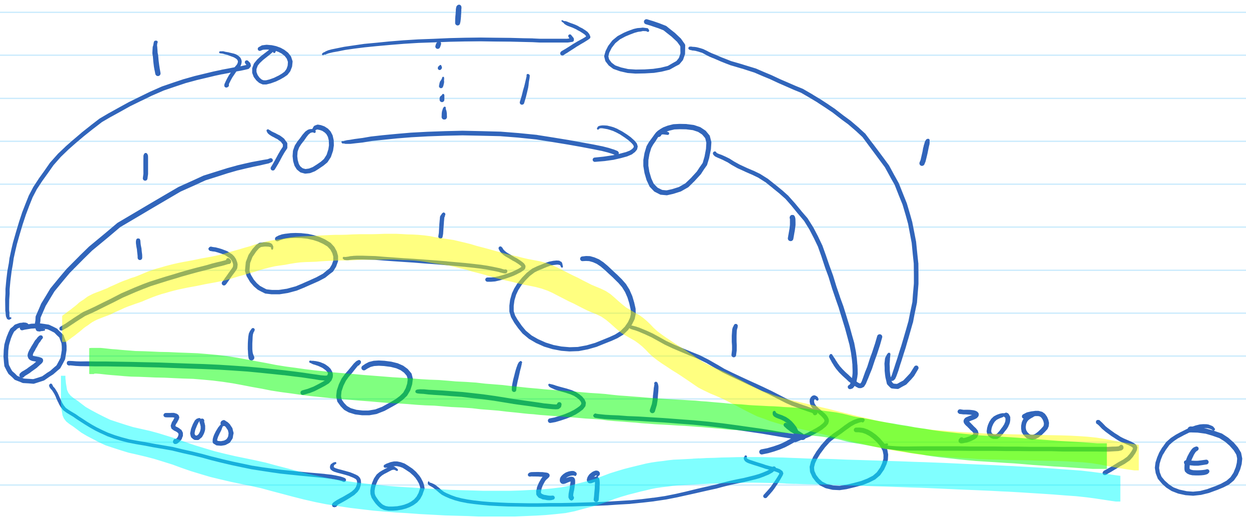
$$f^{out}(v) - f^{in}(v) = 0 \text{ for all } v \in A - \{s\}$$ conservation

$$v(f) = \sum_{v \in A} f^{out}(v) - f^{in}(v)$$ $v \neq s$ terms are 0

$$= \sum_{v \in A} \left(\sum_{(v,x) \in E} f(v,x) - \sum_{(u,v) \in E} f(u,v) \right)$$ def f^{out}, f^{in}

$$= \sum_{v \in A} \left(\sum_{\substack{(v,x) \in E \\ x \notin A}} f(v,x) - \sum_{\substack{(u,v) \in E \\ u \notin A}} f(u,v) \right)$$ if $a, a' \in A$ then $f(a,a')$ appears in both sums and cancels

$$= f^{out}(A) - f^{in}(A)$$ def $f^{out}(A)$, $f^{in}(A)$



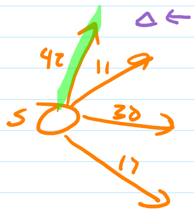
which path should we choose 1st?

Maximum Flow in Polynomial Time

```
MAX-FLOW(G)

    $f(u,v) \leftarrow 0$ for all $(u,v) \in E$
    $G_r \leftarrow G$
    $\Delta \leftarrow 2^{\lfloor \log_2 \max_{(s,v)\in E} c(s,v) \rfloor}$

$O(\log_2 C)$  while $\Delta \geq 1$
# iterations??? →  while there is a path P $s \leadsto t$ in $G_{r,\Delta}$
$O(m)$                  $O(n)$ [ augment(P, f)
                               [ update ($G_r$, P)
                    $\Delta \leftarrow \Delta/2$
```



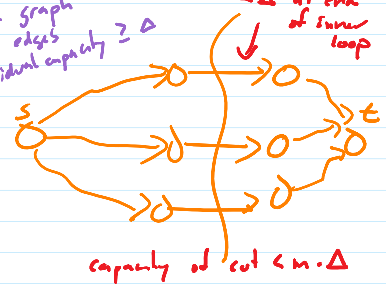
$\Delta \leftarrow 32$ $(32 \leq 42)$

↑ residual graph w/ only edges w/ residual capacity $\geq \Delta$

total $\underbrace{O(\log_2 C \cdot m \cdot n)}$

polynomial in size of graph + # bits in capacities

residual capacity $< \Delta$ at end of inner loop



capacity of cut $< m \cdot \Delta$

INV (outermost loop): there is a cut (A, B) s.t. $c(A,B) \leq v(f) + 2m \cdot \Delta$

Basis:

Maintenance: Let $A^* = \{ v \mid s \leadsto v \text{ in } G_{r,\Delta} \}$, $B^* = V - A^*$

$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$

THM of Mar 27

$$= \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} f(u,v) - \sum_{\substack{(u,v)\in E \\ u \in B^* \\ v \in A^*}} f(u,v)$$

def f^{out}, f^{in}

for edges $A^* \to \underline{B^*}$
$c_r(u,v) = c(u,v) - f(u,v) < \Delta$
(if $c_r(u,v) > \Delta$ then
$(u,v) \in G_{r,\Delta}$ so $s \leadsto u \to v$ $\Rightarrow\Leftarrow$)

$$\geq \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} (c(u,v) - \Delta_{old}) - \sum_{\substack{(u,v)\in E \\ u \in B^* \\ v \in A^*}} \Delta_{old}$$

for edges $\overset{u}{\boxed{B^*}} \to \overset{v}{A^*}$
$f(u,v) < \Delta$
(if $f(u,v) \geq \Delta$
then $c_r(v,u) \geq \Delta$
so $(v,u) \in G_{r,\Delta}$
and $s \leadsto v \to u$ $\Rightarrow\Leftarrow$)

$$\geq c(A^*, B^*) - m\Delta_{old}$$

$$= c(A^*, B^*) - 2m\Delta_{new}$$

any flow $\leq$ capacity of any cut ↓

INV ↓

$$v(f_{new}) \leq v(f^*) \leq c(A^*, B^*) \leq v(f_{old}) + 2m\Delta$$

$$v(f_{new}) - v(f_{old}) \leq \underline{2m\Delta}$$

$m=10$
$\Delta=16$
320

we add at most this during 1 iteration of outer loop

each iteration of inner adds $\geq \Delta$ flow

so $\leq \underline{2m}$ iterations of inner

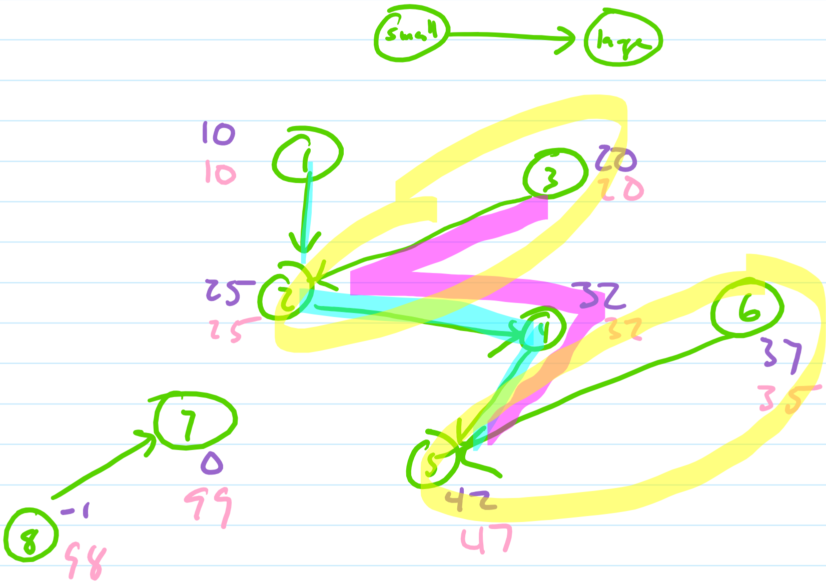
→ record of results of comps between elements

Comparison Graphs

Pre: n≥1 A distinct

```
MAX(A, n)
    max ← A[1]
    for i = 2 to n
        if A[i] > max
            max ← A[i]
    return max
```

n-1 comparisons



Lower Bounds

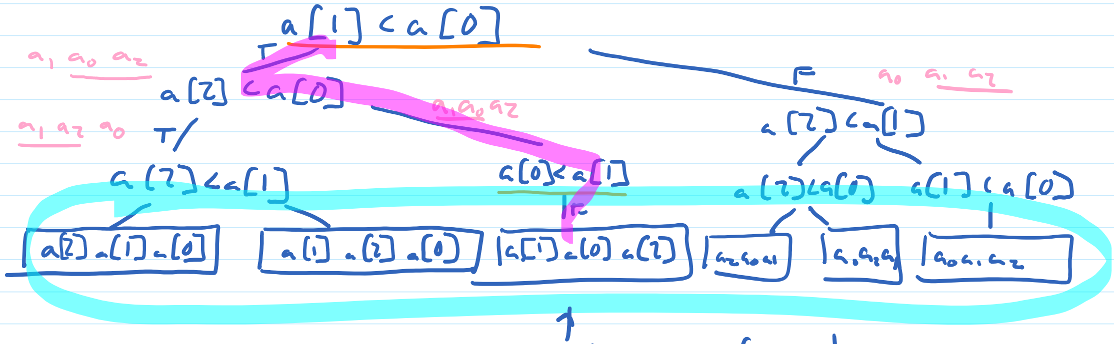
Lower bound for problem P is f(n) means

no alg that solves P has worst-case $o(f(n))$

Lower bound for sorting:

```
for i = 0 to n-2
  for j = 0 to n-2-i
    if (a[j+1] < a[j]) then
      swap(a, j, j+1)
```

Decision Tree



all n! perms of input

DT for ~~bubble~~ any! sort is a binary tree w/ n! leaves

so tree has height $\geq \log_2 n!$

$\approx n \log n$