Vertex Cover and Independent Set

SORT

Alg A : poly time
B CONV HULL
poly time

better than $n \log n$

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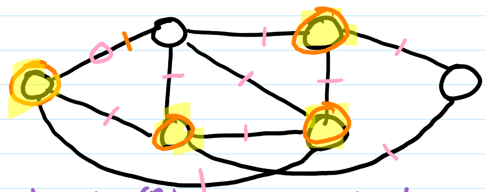
if B easy then A
if A hard then B

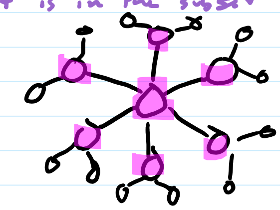
$A \le_p B$

Vertex Cover : Given G and k, determine if there is a vertex cover of size $\le k$

subset of vertices s.t. for each edge, at least one endpoint is in the subset

k = 2 ? NO
k = 3 ? NO
k = 4 ? YES

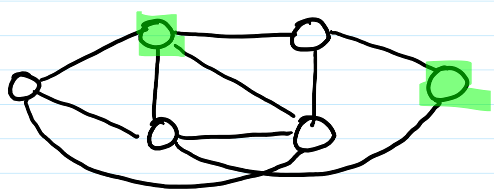




brute force: check all $\binom{n}{k}$ size-k subsets ∴ not polynomial

Independent Set: Given G and k, determine if there is an independent set of size $\ge k$

subset of vertices s.t. no edge between any two verts in subset



Algorithm for VC: return IS(G, n − k)

input : G, k

$VC \le_p IS$

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THM: G has VC of size $\le k$ if and only if G has IS of size $\ge n-k$

Proof : $\Longrightarrow$ Suppose G has a VC of size $\le k$. [want to show G has IS of size $\ge n-k$]

Call the VC of size $\le k$ C

Let $S = C^c$ $(V - C)$ [claim: S is an independent set]

no edges between two verts in S

Let $\boxed{u, v \in S}$ $u \ne v$ [want $(u,v) \notin E$]

Suppose $(u, v) \in E$

Then $u \in C$ or $v \in C$ def. vertex cover

So $\boxed{u \notin S \text{ or } v \notin S}$ $\Rightarrow\Leftarrow$ choice of S

∴ $(u,v) \notin E$

∴ for all $u, v \in S$, $u, v \notin E$

$\therefore (u,v) \notin E$

$\therefore$ for all $u, v \in S$, $u, v \notin E$

$\therefore$ S is an ind set

def ind. set.

$\Leftarrow$: similar

count uses as 1 step

$A \leq_P B$: there is some poly-time alg that solves A that uses a solution for B as a subroutine

A is polynomially reducible to B

there is an alg for B w/ worst case $O(n^k)$ for some $k \in \mathbb{Z}$

THM: If $A \leq_P B$ then if $B \in P$ then $A \in P$

Alg for A: poly time n^3
make n^{10} calls to B n^7 $\quad n^7 \cdot n^{10} = n^{17}$
poly time n^2

$O(n^{17})$

COR: If $A \leq_P B$ then if $A \notin P$ then $B \notin P$

Hamiltonian Path and Long Path and Travelling Salesperson

Hamiltonian Path: Given G, is there a simple path through all vertices?

Long Path: Given G, k, is there a simple path of length $\geq k$?

HP $\leq_p$ LP: HP(G)

let k = size of G.V
return LP(G, k-1)

Travelling Salesperson Optimization: Given complete weighted G, what is the total weight of the min-weight tour?

Decision: Given complete weighted G, and a bound k, is there a tour of total weight $\leq k$?

YES or NO answer

HC $\leq_p$ TSP-DEC:

Hamiltonian Cycle

Alg HC(G): 1) construct input x to TSP-DEC } poly-time
2) output TSP-DEC(x)

[want: given G, find weighted complete G', k s.t. G' has a tour of tot weight $\leq k$ $\Updownarrow$ G has a HC]

if TSP-OPT $\in$ P then TSP-DEC $\in$ P

TSP-DEC(G, k)
min ← TSP-OPT(G)
return min $\leq$ k

TSP-DEC $\leq_p$ TSP-OPT

if VC-DEC $\in$ P then VC-OPT $\in$ P

binary search for max k s.t. VC-DEC(k)=T

Let G' have same vertex set as G

s.t. $w(u,v) = \begin{cases} 0 & \text{if } (u,v) \in G \\ 1 & \text{if } (u,v) \notin G \end{cases}$

if G has HC then G' has a tour of tot. weight 0

if G' has a tour of tot weight ≤ 0 then G has HC

G has HC $\longleftrightarrow$ G' has a tour of tot weight ≤ 0