Examples

Interval Scheduling: Given n requests with start $s(i)$, finish $f(i)$ find largest set of pairwise compatible intervals

↳ non-overlapping

Weighted Interval Scheduling: add weight w_i to each request, find set of compatible intervals that maximizes total weight

machinists welders

Stable Matching: given two groups M, W of size n and preferences for each, find 1-1 onto fxn $M \to W$ s.t. no pair wants to make a unilateral change

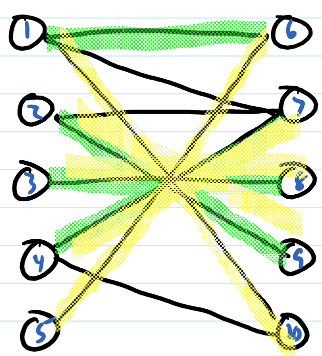
ordering of members of other group

no (m,w) not paired w/ each other who would rather be w/ each other than who they are paired with

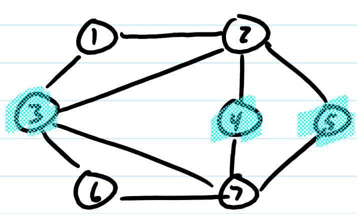
Bipartite Matching: Given bipartite G, find maximum matching

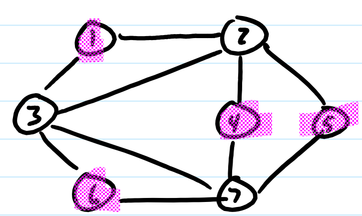
largest set of edges s.t. each vert has ≤ 1 edge

verts split into two parts s.t. no edges within same part



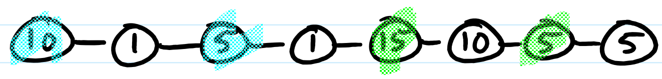
Independent Set: Given graph G, find largest set of vertices s.t. no edges between vertices in the set





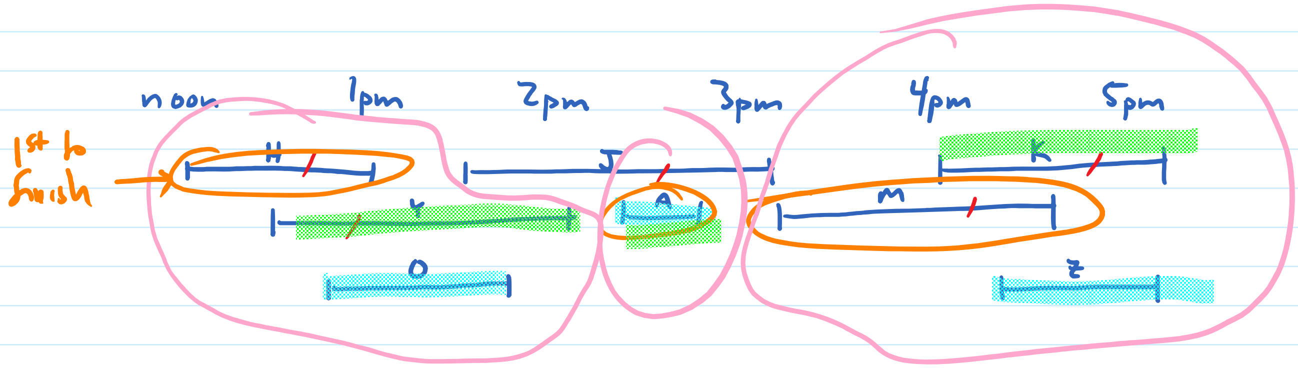
NP-complete

Competitive Facility Location: Given G with weighted verts, bound B
game between P1, P2 alternating choosing
vert s.t. not adjacent to already chosen,
is there a strategy for P2 to guarantee
a total $\geq B$?



I win 20 - 15

Continuing Education Credits



8 sessions - pick maximum number of compatible (nonoverlapping) sessions

Interval Scheduling

Given intervals labelled $1, \dots, n$ w/ interval i's start, finish = $s(i)$, $f(i)$,
find largest set of pairwise compatible intervals

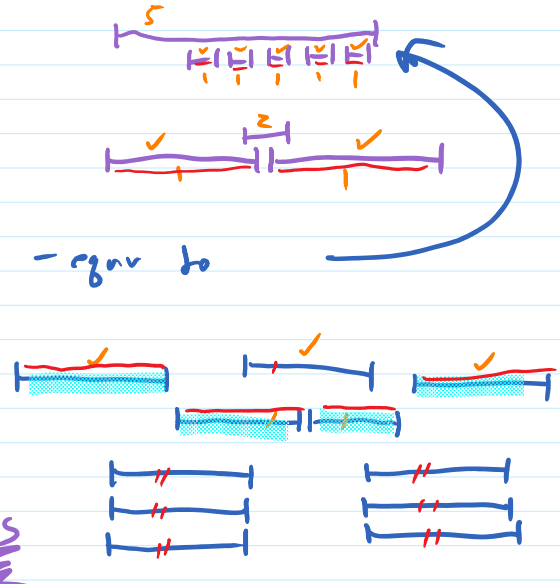
Greedy : choose 1st to start? NO

choose shortest? NO

choose last to finish NO - equiv to

choose fewest overlaps

choose 1st to finish YES



Precondition: R sorted by ↑ finishing time

```
k ← 0
R ← {1,...,n}            indices of intervals compatible w/A
A ← ∅                    our selected intervals
while R ≠ ∅              while intervals left to choose from
   choose i∈R to minimize f(i)      choose one that finishes first
   A ← A ∪ {i}                      update selected
   remove from R all incompat w/ i  throw out intervals overlapping selected
   k ← k+1
return A
```

Let Θ be opt sol $= j_1, \dots, j_m$ in order of ↑ finish
so $s(j_1) \le f(j_1) \le s(j_2) \le f(j_2) \le \dots \le f(j_m)$

INVARIANT : after iteration k
a) A is pairwise compatible and sorted by ↑ finish time
b) $f(a_k) \le f(j_k)$ or $k = 0$ — on par with or ahead of optimal soln Θ
c) $|A| = k$
d) R is intervals ℓ s.t. $s(\ell) \ge f(a_k)$ (or $k=0$ and $R = A$)

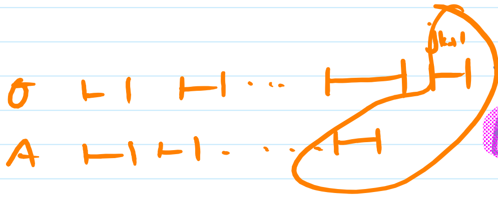
Initialization trivial

Maintenance : Suppose that INV is true after k iterations and $R \ne \emptyset$

b) $f(j_{k+1}) > f(j_k)$ — Θ sorted

$s(j_{k+1}) \ge f(j_k)$ — Θ is a solution (otherwise j_k, j_{k+1} overlap)

j_{k+1} $f(a_k) \le f(j_k)$ — Invariant (otherwise j_k, j_{k+1} overlap)



$f(a_k) \le f(j_k)$ Invariant

[want $f(a_{k+1}) \le f(j_{k+1})$]

j_{k+1} is compatible with A

Termination:

Post-condition:

Interval Scheduling

Precondition: $f(1) \le f(2) \le \cdots \le f(n)$ intervals ordered by ↑ finish
$s(1) < f(1), \ldots, s(n) < f(n)$ start of any interval is before its end

$k \leftarrow 0$
$R \leftarrow \{1, \ldots, n\}$ indices of intervals compatible with A
$A \leftarrow \emptyset$ our selected intervals
while $R \neq \emptyset$ while intervals left to choose from
  choose $i \in R$ to minimize $f(i)$ choose one that finishes first
  $A \leftarrow A \cup \{i\}$ update selected
  remove from R those incompatible w/ i throw out intervals overlapping selected
  $k \leftarrow k+1$
return A

Let $\mathcal{O}$ be an opt sol $= j_1, \ldots, j_m$ in order of ↑ finish
so $s(j_1) < f(j_1) \le s(j_2) \le f(j_2) \le \cdots \le f(j_m)$

INVARIANT: after iteration k
a) A is pairwise compatible and sorted by ↑ finish time
b) $f(a_k) \le f(j_k)$ or $k=0$ on pace with or ahead of optimal soln $\mathcal{O}$
c) $|A| = k$
d) R is intervals ℓ s.t. $s(\ell) \ge f(a_k)$ (or $k=0$ and $R=A$)
e) $A, R \subseteq$ original R

Initialization:
a) $A = \emptyset$, so vacuously true
b) $k=0$
c) $|A| = 0$ $k=0$
d) $k=0$ and $R=A$
e) $R =$ original R, $A = \emptyset \subseteq$ original R

Maintenance: Suppose that INV is true after k iterations and $R \neq \emptyset$

a) $i \in R$ choice of i

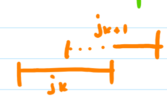
$f(i) > s(i) \ge f(a_k) > f(a_{k-1}) > \cdots > f(a_1)$ precondition, INV d, INV a

i is compatible w/ each elt of A_{old} the above shows no overlap

elts of A_{old} were pairwise compatible INV a

$A_{new} = A_{old} \cup \{i\}$ is pairwise compatible and sorted by ↑ finish above 2 lines cover all pairs

b) $f(j_{k_{old}+1}) > f(j_{k_{old}})$ $\mathcal{O}$ sorted



$s(j_{k_{old}+1}) \ge f(j_{k_{old}})$ if $s(j_{k+1}) < f(j_k)$ then j_k, j_{k+1} overlap and so $\mathcal{O}$ is not a solution

$f(a_{k_{old}}) \le f(j_{k_{old}})$ INV b

$f(a_1) < \cdots < f(a_{k_{old}})$ INV a

$j_{k_{old}+1} \in R$ INV d (have $s(j_{k+1}) \ge f(j_k) \ge f(a_k)$)

$f(i) \le f(j_{k_{old}+1})$ choice of i — if $i \neq j_{k+1}$ then i is an interval with an even earlier start time

$f(a_{k_{old}+1}) = f(i)$ i is chosen as the next elt of A

$f(a_{k_{new}}) \leq f(j_{k_{new}})$ substitution (also $k_{new} = k_{old}+1$)

c) every time through the loop, k goes up by 1 and one new element of A is added (we know it is new b/c by INV a, d + PRE we have $f(a_1) < \cdots < f(a_k) \leq s(i) < f(i)$, so $i \neq a_j$ for all j

d) $R_{old} = \{ \ell \mid s(\ell) \geq f(a_{k_{old}}) \}$ INV d

$R_{new} = R_{old} - \{ \ell \mid \ell \in R_{old} \text{ and } s(\ell) < f(a_{k_{new}}) \}$ code (those incompatible w/ $i = a_{k_{new}}$ are those that start before it finishes)

$= \{ \ell \mid s(\ell) \geq f(a_{k_{old}}) \text{ and } s(\ell) \geq f(a_{k_{new}}) \}$

$= \{ \ell \mid s(\ell) \geq f(a_{k_{new}}) \}$ $s(\ell) \geq f(a_{k_{new}}) \rightarrow s(\ell) \geq f(a_{k_{old}})$ b/c $f(a_{k_{new}}) > f(a_{k_{old}})$

e) We never add anything to R
We only add to A things that are in a subset of the original R

Termination: $|A| = k$ and $A \subseteq$ original R, so loop must terminate at least when $|A| = |\text{original } R|$

Post-condition: INV a shows that A is a valid solution

$m = |O| \geq |A| = k$ otherwise O isn't optimal

$R = \emptyset$ guard is false

$k = |A| \geq |O| = m$ otherwise $s(j_m) \geq f(j_{m-1}) > \cdots > f(j_k) \geq f(a_k)$ by INV b
so $j_m \in R$ by INV d
which contradicts $R = \emptyset$

$|A| = |O|$ each is both $\leq$ and $\geq$ the other

so A is optimal   same size as any other optimal solution