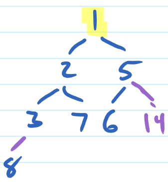
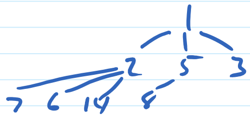
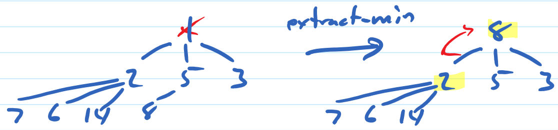
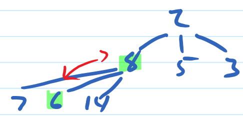
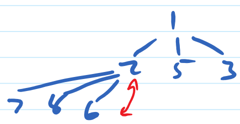
d-ary Heaps











	d-ary heap	Dijkstra
extract-min:	$d \cdot \log_d n$	n
decrease_key:	$\log_d n$	m

$n\left(\frac{2m}{n} - 1\right)$

pick $d = \max\left(\frac{m}{n}, 2\right)$

total: $\underbrace{\cancel{n}\cdot\frac{m}{\cancel{n}}\cdot\log_{\frac{m}{n}} n}_{\text{extract-min}} + \underbrace{m\cdot\log_{\frac{m}{n}} n}_{\text{decrease-key}}$

$O\left(m\cdot \underbrace{\log_{\frac{m}{n}} n}\right) = O(n \log n)$ if $m \approx c\cdot n$ (sparse)

$= O(n^2)$ if $m \approx n^2$ (dense)

$\log_{\frac{c\cdot n}{n}} n = \log_c n$ for sparse case

Invariant

Dijkstra (G, ℓ)

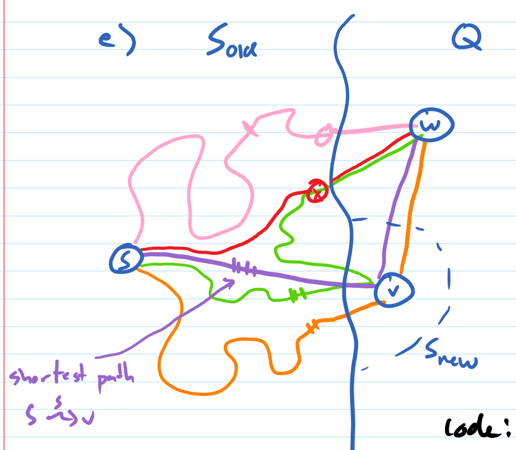
$Q \leftarrow \emptyset$
for each vertex v
  $d'[v] \leftarrow \infty$
  $\pi[v] \leftarrow NIL$
$d'[s] = 0$
for each v
  enqueue$(Q, v, d'[v])$

while $Q \neq \emptyset$
  v = extract_min(Q)
  $d[v] \leftarrow d'[v]$
  $S \leftarrow S \cup \{v\}$
  for $(v,w) \in E$ where $w \in Q$
    if $d[v] + \ell(v,w) < d'(w)$
      $d'[w] = d[v] + \ell(v,w)$
      $\pi[w] = v$
      decrease_priority$(Q, w, d'[w])$

INVARIANT

a) $|S| \geq 1 \rightarrow s \in S$
b) S, Q partition V
c) $|Q| = n -$ # iterations of loop
d) for $v \in S$, $d[v] = \delta(s,v)$ — total weight of shortest path $s \leadsto v$
  $\pi[v]$ = next-to-last on that path
e) for $v \in Q$, $d'[v]$ = total weight of shortest path $s \leadsto v$ using intermediate vertices in S
  $\pi[v]$ = next-to-last on that path
f) for all v, $d'[v]$ is the priority of v in Q
g) for all v, $\pi[v] = NIL$ or $\pi[v] \in S$

e) S_{old} Q



before: $d'[w]$ = min tot weight of paths $s \overset{S}{\leadsto} v$

code: $d'_{new}[w] = \min(d'_{old}[w], \quad)$

$d'_{new}[w] \leq d'_{old}[w]$ (def min)
$d'_{new}[w] \leq \ell(\quad)$

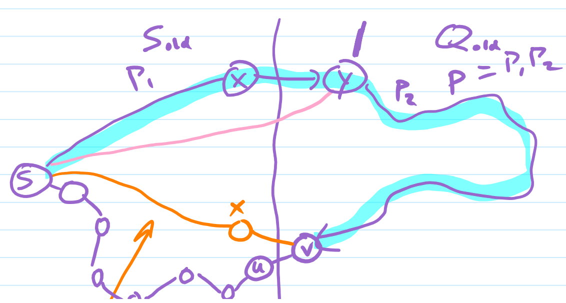
let P be any path $s \leadsto v$

$\ell(\quad) \geq d'_{old}[w] \geq d'_{new}[w]$
INV e   min

$\ell(\quad) \geq \ell(\quad) \geq d'_{new}[w]$
replace $s \leadsto v$ with $s \leadsto v$ which is shortest using int verts in S min

$\ell(\quad) = \ell(s \leadsto x) + w(x,w)$
INV e $\rightarrow \geq \ell(s \leadsto x) + w(x,w)$
min $\rightarrow \geq d_{old}[w]$

d) $d[v]$ is total weight of a shortest path $s \leadsto v$ (for all $v \in S$)
(min tot weight)



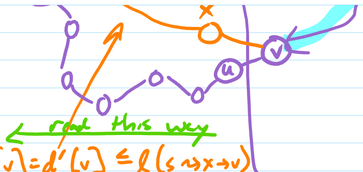
$d_{new}[v] = d'[v]$
$d'[v] \leq d'[y]$ (v chosen over y by extract_min)
$d'[y] \leq \ell(P_1)$ (INV e + P_1 uses intermediate vertices from S)
$\ell(P_1) \leq \ell(P_1) + \ell(P_2)$ (no neg edges)

2 kinds of paths

$s \rightsquigarrow v$

those using intermediate verts in S_{old}

and others



read this way

$d_{new}[v] = d'[v] \le \ell(s \rightsquigarrow x \to v)$

INV e

vertices from s)

$\ell(P_1) \le \ell(P_1) + \ell(P_2)$ (no neg edges)

$= \ell(P)$

$d_{new}[v] = d'[v]$ is weight of a path $s \rightsquigarrow v$ (INV e)

$d_{new}[v] \le \ell$ (any other path $s \rightsquigarrow v$)

$\therefore$ $d_{new}[v]$ is weight of shortest path $s \rightsquigarrow v$

Minimum Steiner Tree

https://www.daysofwonder.com/tickettoride/en/usa/

Minimum Steiner Tree

positive weights   connected

Given weighted, undirected G and a subset of vertices A,
find total weight of set of edges that connect A
while minimizing total weight

HARD! (NP-complete)

Special cases: $|A| = 2$ shortest path → Dijkstra

$A = V$ minimum spanning tree ↓

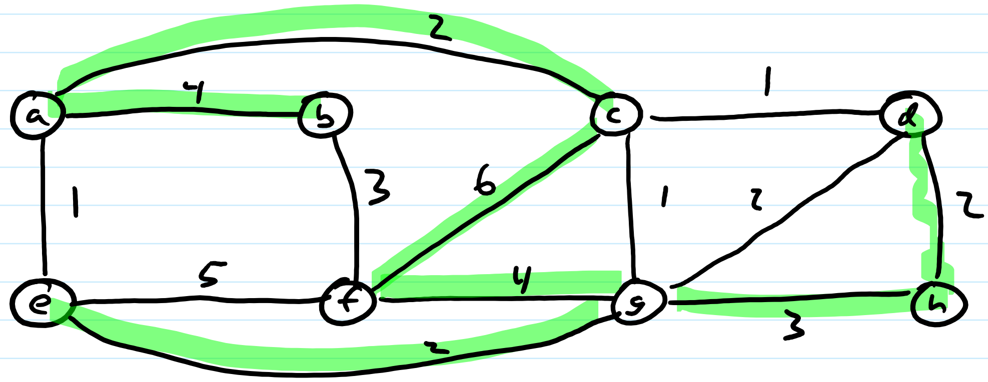
finding set of edges that connects all vertices with min total weight

TTR which 2 cards to keep (s_1, d_1) (s_2, d_2) (s_3, d_3)?

which 2 minimize $\min(\text{Dijkstra}(s_i, d_i) + \text{Dijkstra}(s_j, d_j),$

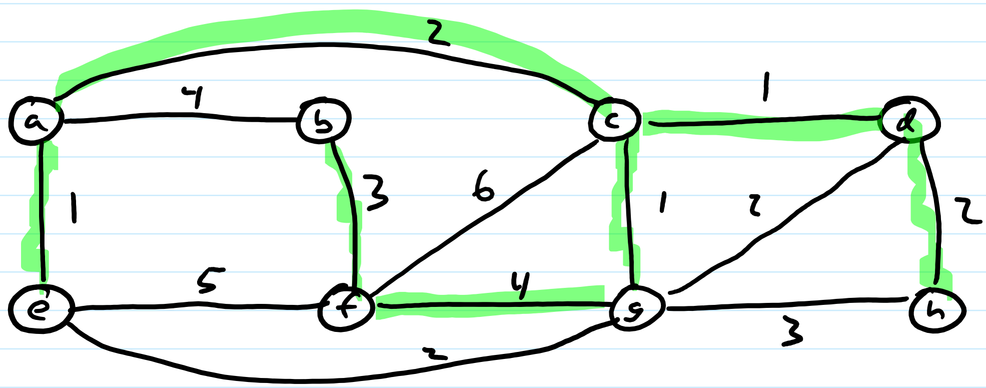
$\text{MinSteinerTree}(s_i, s_j, d_i, d_j))$?

Prim's Algorithm



tot weight

23



weight 13

Prim ~~Dijkstra~~ (G, ℓ)

$Q \leftarrow \emptyset$

for each vertex v

$\quad d'[v] \leftarrow \infty$

$\quad \pi[v] \leftarrow$ NIL

$d'[s] = 0$ choose s arbitrarily

for each v

$\quad$ enqueue$(Q, v, d'[v])$

while $Q \neq \emptyset$

$\quad v = $ extract_min(Q)

$\quad \cancel{d[v] \leftarrow d'[v]}$

$\quad S \leftarrow S \cup \{v\}$

$\quad$ for $(v, w) \in E$ where $w \in Q$

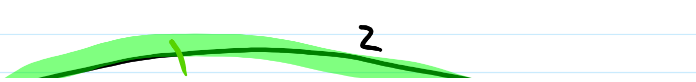
$\quad\quad$ if $\cancel{d[v]} + \ell(v,w) < d'(w)$

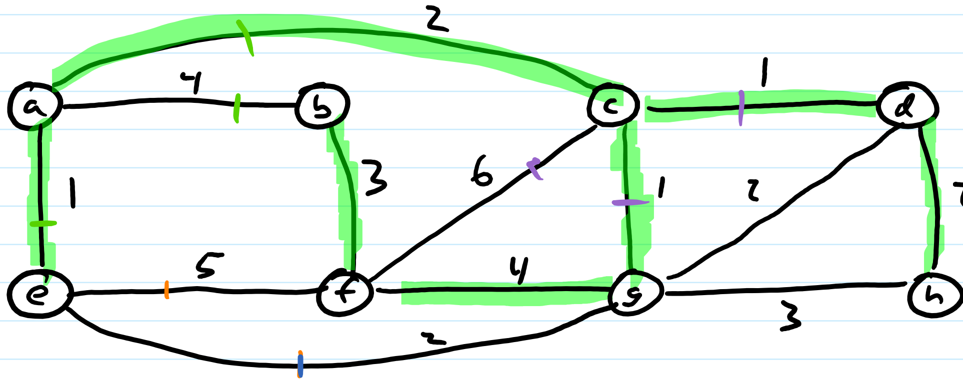
$\quad\quad\quad d'[w] = \cancel{d[v]} + \ell(v,w)$ $\quad d'[w] =$ min weight of edges connecting w to S $= \min_{u \in S} \ell(u,w)$

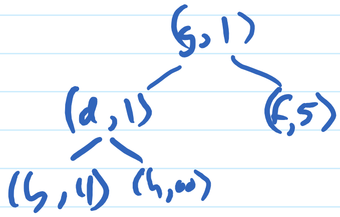
$\quad\quad\quad \pi[w] = v$

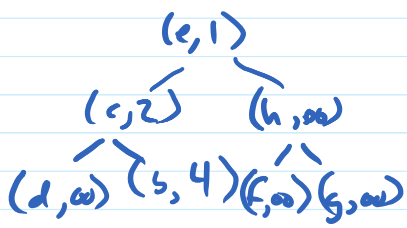
$\quad\quad\quad$ decrease_priority$(Q, w, d'[w])$

return $T = \{ (v, \pi[v]) \mid v \neq s \}$









~~d~~	~~1~~
~~f~~	~~5 4~~
b	~~4~~ 3
~~h~~	~~∞ 3 2~~