Vendor Scheduling

	Week 1	2	3	4	5
Brooklyn	6	3	10	4	1
Stratford	4	6	3	8	20

$M = 10$

net BBBSS $= 47 - 10 = 37$

net profit $= 41$

brute force 2^n sched to check

SSSSS 21 ← not opt for 4 weeks

SSBB $24 - 10 = 14$ is opt for 4 week given end in S

BBBB 23

so add parameter

		6	3	10	4	1
		4	6	3	8	20
OPT_B	0	6	9 B	19 B	23 B	24 B
OPT_S	0	4	10 S	13 S	24 B	44 S

$m = 3$

max(4, 9)

max(9, 10)

max(19+8−3, 13+8)

max(23+1, 24+1−3) = 24

$\boxed{\text{B B B S S}}$ opt schedule

$OPT_B(i)$ = max profit for weeks $1 \ldots i$ ending at week i at B

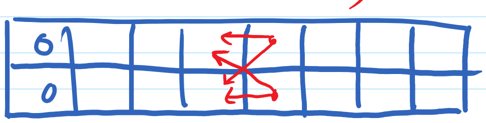
$OPT_S(i)$ = " " S

$$OPT_B(i) = \begin{cases} 0 & \text{if } i = 0 \\ \max\left(OPT_B(i-1),\ OPT_S(i-1) - M\right) + B_i \end{cases}$$

$\Theta(1)$

$$OPT_S(i) = \begin{cases} 0 & \text{if } i = 0 \\ \max\left(OPT_S(i-1),\ OPT_B(i-1) - M\right) + S_i \end{cases}$$

max profit → OPT_B, OPT_S



$\Theta(n)$ entries

$\Theta(1)$ time per entry

$\Theta(n)$ total

B	0	6 B	9	19	23	24
S	0	4 S	10	13	24 S	44 B

max(13+8, 19+8−3)

so week 3 was in B

B B B S S

All Pairs Shortest Paths

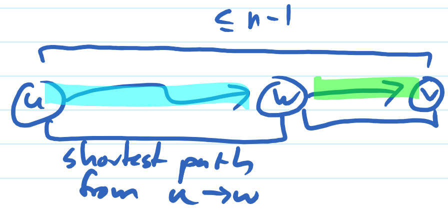
→ possibly negative (but no neg-weight cycles [so if neg weights, then directed])

Given weighted G, find shortest path between all pairs of vertices

$O(n^2 \log n)$ if $m \in \Theta(n)$ [sparse graphs, no neg-weight edges: run Dijkstra from each vertex
sparse graphs, neg-weight edges : see Johnson's alg

$$d[u,v] = \begin{cases} 0 & \text{if } u = v \\ \min_{w \in V} d[u,w] + \ell(w,v) & \end{cases}$$

what's the next-to-last vertex



$d[u,v_1]$
$d[u,v_2]$
$d[u,v_1]$
⋮

circular references
subproblem not getting smaller

$d[u,v,k]$ = tot weight of shortest path $u \to v$ that uses $\le k$ edges

$$= \begin{cases} 0 & \text{if } u=v \\ \infty & \text{if } u \ne v, k=0 \\ \min_{w \in V} d[u,w,k-1] + \ell(w,v) & \end{cases}$$

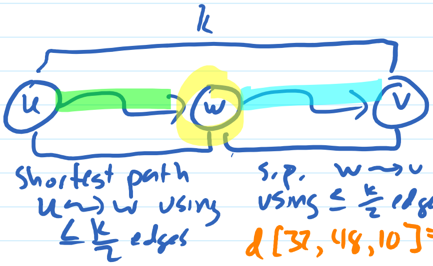
optimize over all choices of next-to-last

$\Theta(n^3)$ entries $\Theta(n)$ time each $\Theta(n^4)$ overall

25 vertices
$d[u,v,32]$
16
8
⋮
$\log_2 n$ values of k to compute for

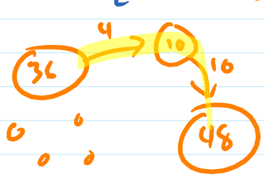
what's the middle vertex

$$d[u,v,k] = \begin{cases} 0 & \text{if } u=v \text{ and } k=0 \\ \infty & \text{if } u \ne v \text{ and } k=0 \text{ or } (u,v) \notin E \text{ and } k=1 \\ \ell(u,v) & \text{if } (u,v) \in E \text{ and } k=1 \\ \min_{w \in V} \left(d[u,w,\tfrac{k}{2}] + d[w,v,\tfrac{k}{2}] \right) & \text{otherwise.} \end{cases}$$

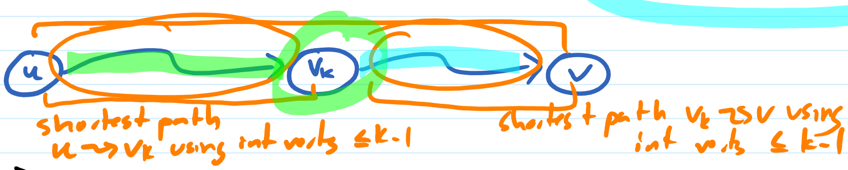


$\Theta(n^2 \cdot \log_2 n)$ $\Theta(n)$ time each
$\Theta(n^3 \log n)$

$d[37,48,10] = 14$ $d(37,48,9) = \infty$



$d[u,v,k]$ = tot weight of shortest path $u \to v$ using intermediate vertices numbered $\le k$



Floyd-Warshall

$$d[u,v,k] = \begin{cases} 0 & \text{if } u=v \\ \ell(u,v) & \text{if } u \ne v \text{ and } k=0 \text{ and } (u,v) \in E \\ \infty & \text{if } u \ne v \text{ and } k=0 \text{ and } (u,v) \notin E \\ \min\left(\underbrace{d[u,v_k,k-1] + d[v_k,v,k-1]}_{\text{use vertex } k},\ \underbrace{d[u,v,k-1]}_{\text{don't use vertex } k} \right) & \end{cases}$$

$\Theta(n^3)$ entries
$\Theta(1)$ entry

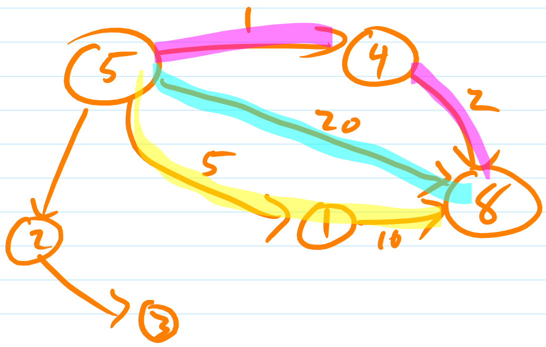
$\Theta(n^3)$ total

use vertex k

don't use vertex k

$\Theta(1)$ entry

$\Theta(n^3)$ total ☺



$d[5,8,0] = 20$

$d[5,8,1] = 15$

$d[5,8,2] = 15$

$d[5,8,3] = 15$

$d[5,8,4] = d[5,4,3] + d[4,8,3]$

$= 1 + 2$

$= 3$