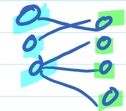
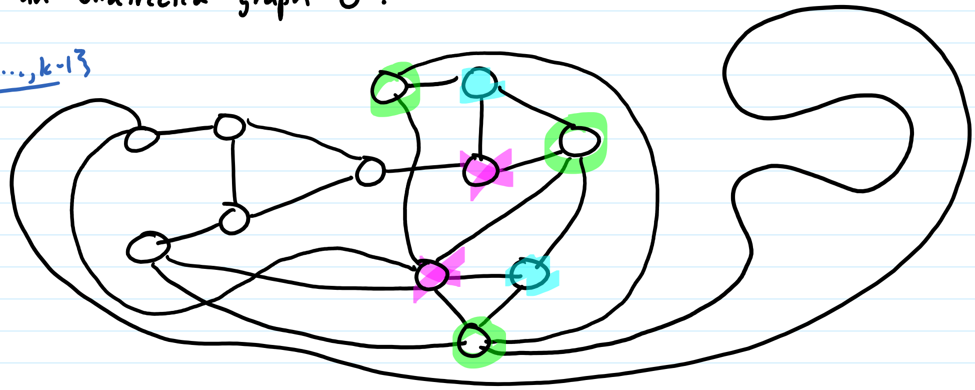
3-COLORING

2-colorable ⟷ bipartite



Coloring of an undirected graph G :

assignment of integer $\{0, \dots, k-1\}$ to each vertex
s.t. $(u,v) \in E \rightarrow c(u) \neq c(v)$



Prove: 3-COLORING is NP-complete

1) 3-COLORING $\in$ NP

3-COLOR-VERIFY(G, c) ← 1 entry for each vertex (poly size)
- if $c(v) \in \{0,1,2\}$
  - for each edge (u,v) — poly iterations
    - if $c(u)=c(v)$ ← poly time
      - output NO
  - output YES
- output NO

2) 3-SAT $\leq_p$ 3-COLORING

goal: given φ, construct G s.t. φ is satisfiable ⟷ G is 3-colorable

3-SAT(φ)
build G poly
return 3-COLOR(G)

C_1 C_2

$(x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee \neg y)$

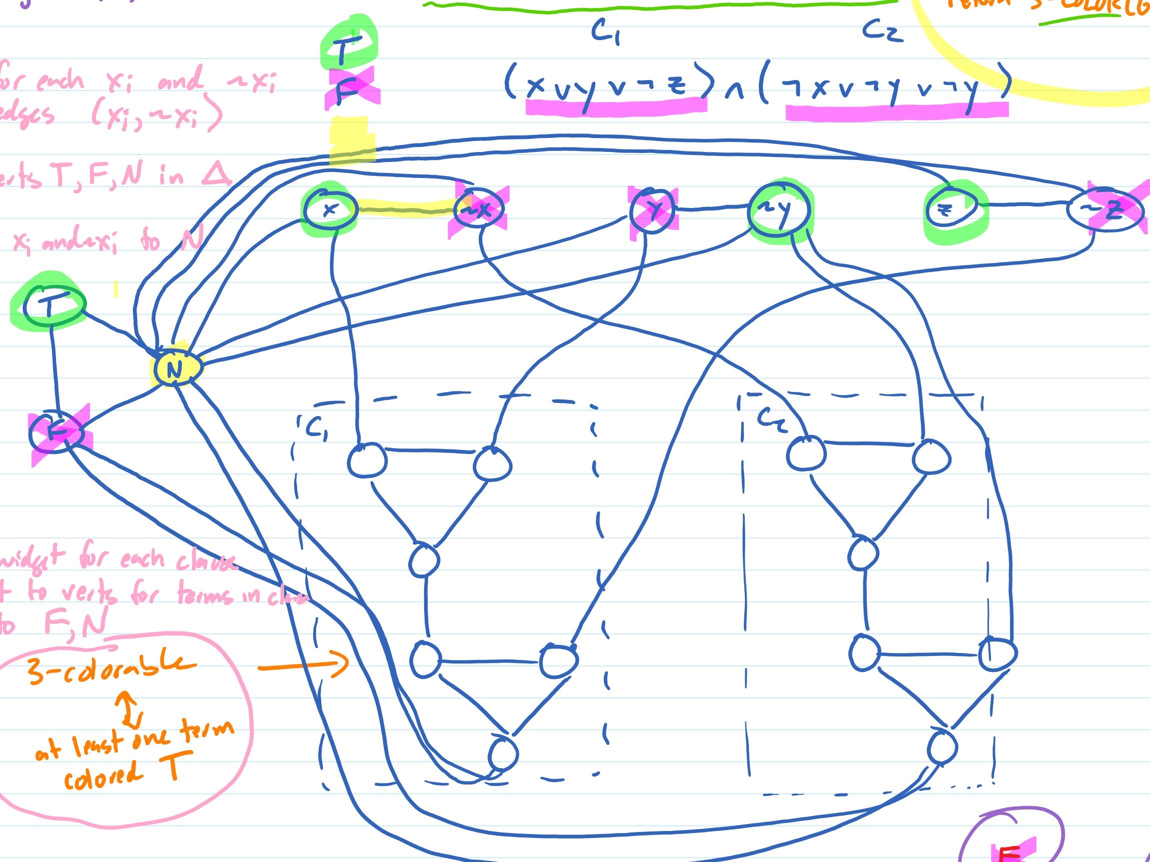
var for each x_i and $\neg x_i$
add edges $(x_i, \neg x_i)$

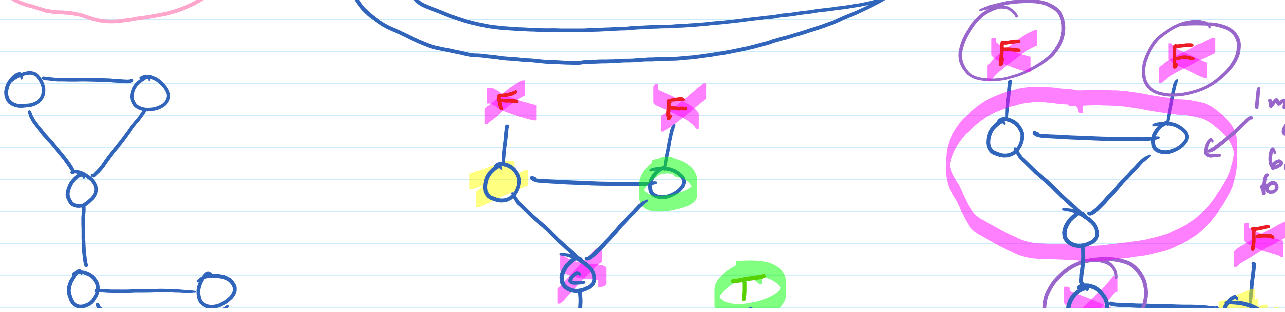
add verts T, F, N in △

connect x_i and $\neg x_i$ to N

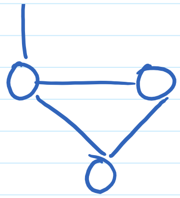
add widget for each clause
connect to verts for terms in clause
and to F, N

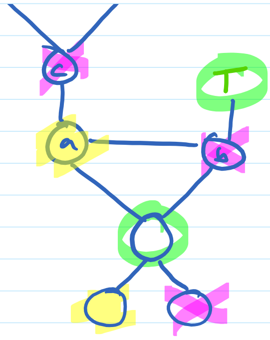
3-colorable ⟷ at least one term colored T

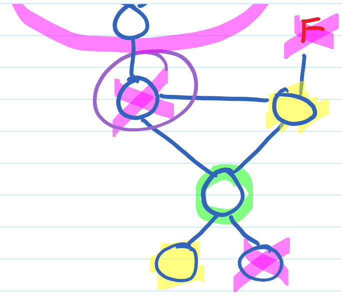




1 must be colored F but all connected to something F




T
a
b

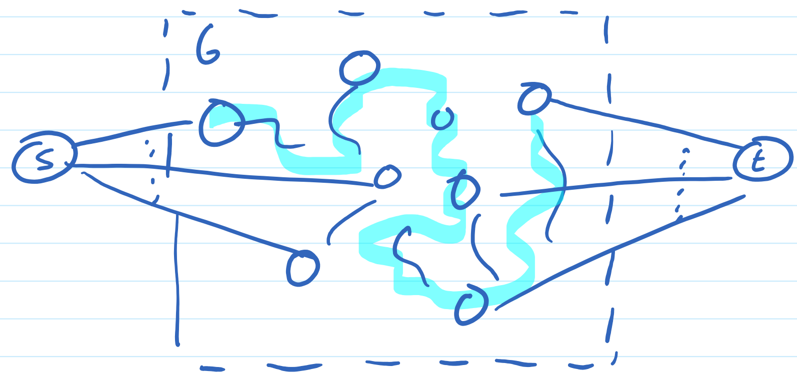

F

s-t HP: Given G, s, t, is there a Hamiltonian path that starts at s, ends at t?

s-t HP $\in$ NP: evidence would be the $s \rightsquigarrow t$ Ham path

HP $\leq_m$ s-t HP : [given G, create G' + pick s, t s.t. G has HP $\Updownarrow$ G' has HP from $s \rightsquigarrow t$]

G'

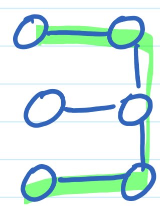


G has HP $v_1, \dots, v_n$
$\downarrow$
G' has $s \sim t$ HP
$s, v_1, \dots, v_n, t$

G' has $s \sim t$ HP
$s, v_1, \dots, v_n, t$
$\downarrow$
$v_1, \dots, v_n$ is HP in G

ALMOST-HP: Given G, is there a path that visits each vertex exactly once, except perhaps misses one

ALMOST-HP is NP-complete
1) in NP
2) for all $X \in NP$, $X \leq_p$ ALMOST-HP
$X \leq_m Y$
$X \leq_p HP \leq_p$ ALMOST-HP

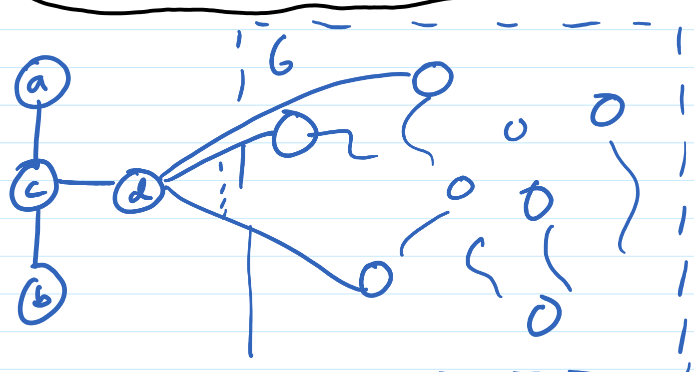


ALMOST-HP $\in$ NP : evidence is the path that's almost an HP

HP $\leq_p$ ALMOST-HP : [given G, create G' s.t. G has HP $\leftrightarrow$ G' has almost-HP]

HP(G)
construct G'
return AHP(G')

G'



G has HP $v_1, \dots, v_n$
$\downarrow$
$a, c, d, v_1, \dots, v_n$ is an AHP in G'

G' has ALMOST-HP
$\downarrow$
AHP is $a, c, d, v_1, \dots, v_n$
or $b, c, d, v_1, \dots, v_n$

Karp reduction: 1 call
return result of that call

or $b, c, d, v_1, \ldots, v_n$

HP in G

Karp reduction: 1 call
return result of that call

X is NP-complete and $X \in P$ $\longrightarrow$ $P = NP$ $1M Millenium prize

$X \leq_P Y$ and $Y \in P$ then $X \in P$

S,

X(x)

⋮

Y(y_1)

⋮

Y(y_2)

for i=1 to n

Y(y_n)

⋮

polynomial time

+

polynomial # calls to Y

if $Y \in P$ then this is a polynomial time solution for X

$X \in P$

Y is new problem

X is NP-complete prob

$Y \leq_P X$ shows
you can write bad alg for Y

Chain of Reductions

To show X is NP-complete: 1) show $X \in NP$
2) show $Y \le_p X$ for some NP-complete Y

INDEPENDENT SET
$\le_p$ VERTEX COVER $\le_p$ INDEPENDENT SET

$Y \le_p$ SAT $\le_p$ 3-SAT $\le_p$ DIRECTED HAMILTONIAN CYCLE $\le_p$ HAMILTONIAN CYCLE $\le_p$ HAMILTONIAN PATH

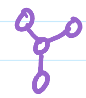
$\le_p$ 3-COLOR

$\le_p$ TRAVELLING SALESPERSON

Cook-Levin Theorem: SAT is NP-complete
SAT $\in$ NP

for all $Y \in NP$, $Y \le_p$ SAT

SORTED $\le_p$ HP

SORTED(A)
if $A[0] \le A[1] \cdots \le A[n-1]$
return HP()
else return HP()

Non-Decision Problems

$c: V \to \{0,1,2,NIL\}$ $\quad$ $c(v) \neq NIL \to c'(v) = c(v)$

EXTEND-3-COLOR: Given G and a partial coloring c, can c be extended to a valid 3-coloring c'?

NP-complete (reduce from 3-COLORING)

FIND-3-COLOR $\le_p$ EXTEND-3-COLOR

```
FIND-3-COLOR(G)
→ c(v) ← NIL
  if EXTEND-3-COLOR(G,c)=NO then output NO
  for each v
       c_R ← c with c(v)=R
       c_G ← c with c(v)=G
       c_B ← c with c(v)=B
       c ← c_i s.t. EXTEND-3-COLOR(G,c_i)=YES
  return c
```

FIND-3-COLOR $\le_p$ 3-COLORING

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FIND-3-COLORING(G)

  if 3-COLOR(G)=NO return NIL

  c(v) ← NIL for all v
  G' ← G
  add Δ r,g,b

  for each vertex v in G
       G_R ← G' with (v,g) (v,b) added
       G_G ← G' with (v,r) (v,b) added
       G_B ←

       find i s.t. G_i s.t. 3-COLORING(G_i)=YES
       c(v)=i
       G' ← G_i
  return c
```