Dealing with hard problems

$$P \subseteq \underbrace{NP \subseteq PSPACE}_{\substack{\text{most people think} \\ \text{NP-complete and} \\ \text{PSPACE-complete} \\ \text{problems are hard} \\ \text{(not in P)}}} \subseteq \underbrace{EXPTIME}_{\substack{\text{EXPTIME-complete} \\ \text{problems \underline{are} hard}}},$$

Small n? $2^{50} \approx 10^{15}$ but $1.3^{50} \approx 500{,}000$

dynamic programming ↓

Special cases? Hamiltonian Cycle, Average Path Length on directed <u>acyclic</u> graphs $\in P$

bipartite? bounded degree? bounded weights?

Randomized algorithm? $X(x) = YES \rightarrow A(x) = YES$ with probability $\geq f(|x|)$

$X(x) = NO \rightarrow A(x) = NO$ with probability 1

Approximation algorithm? FIND-VERTEX-COVER: given G, output smallest vertex cover C^*

APPROX-VERTEX-COVER: given G, output vertex cover C s.t. $\underbrace{|C| \leq 2 \cdot |C^*|}_{\text{a 2-approximation}}$

k-approximation: output result $\leq k \cdot$ optimal (for a minimization problem)

or $\geq \frac{1}{k} \cdot$ optimal (for a maximization problem)

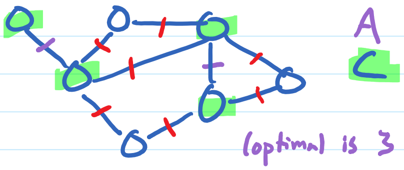
APPROX-VERTEX-COVER

```
GREEDY-VERTEX-COVER(G)
    A ← ∅          edges selected so far
    C ← ∅          the vertex cover
    E' ← E         the edges still uncovered by C

    while E' ≠ ∅
        pick any (u,v) ∈ E'
        A ← A ∪ {(u,v)}
        C ← C ∪ {u,v}
        E' ← E' − edges incident on either u or v
    return C
```

a polynomial-time 2-approximation algorithm

$$|C| \le 2\cdot|C^*|$$

vertex cover output → $|C|$; $|C^*|$ ← smallest vertex cover



(optimal is 3 vertices)

INVARIANT

a) $|A| = k$ number of iterations

b) $E' \subseteq E - A$

c) $|C| = 2\cdot|A|$

d) no two edges in A share an endpoint

e) edges in E', edges in A don't share endpoints

f) C covers A

At termination, $|C^*| \ge |A|$ (INV d → any cover must include one endpoint for each edge in A)

$|C| = 2\cdot|A|$ (INV c)

$$|C| = 2\cdot|A| \le 2\cdot|C^*|$$

APPROX-VERTEX-COVER

GREEDY-VERTEX-COVER(G)

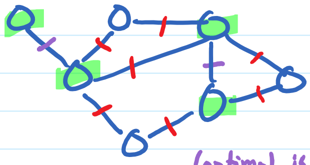
```
$A \leftarrow \emptyset$        edges selected so far
$C \leftarrow \emptyset$        the vertex cover
$E' \leftarrow E$       the edges still uncovered by C

while $E' \neq \emptyset$
    pick any $(u,v) \in E'$
    $A \leftarrow A \cup \{(u,v)\}$
    $C \leftarrow C \cup \{u,v\}$
    $E' \leftarrow E'$ − edges incident on either u or v
return C
```

a polynomial-time 2-approximation algorithm

$|C| \leq 2 \cdot |C^*|$

vertex cover output

smallest vertex cover



A

🟩 C

E' = unmarked edges

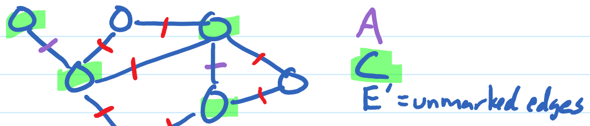
(optimal is 3 vertices)

APPROX-VERTEX-COVER

```
GREEDY-VERTEX-COVER(G)
    A ← ∅          edges selected so far
    C ← ∅          the vertex cover
    E' ← E         the edges still uncovered by C

    while E' ≠ ∅
        pick any (u,v) ∈ E'
        A ← A ∪ {(u,v)}
        C ← C ∪ {u,v}
        E' ← E' − edges incident on either u or v
    return C
```

a polynomial-time 2-approximation algorithm

vertex cover output ↗ $|C| \le 2\cdot|C^*|$ ↖ smallest vertex cover



(optimal is 3 vertices)

INVARIANT

a) $|A| = \boxed{k}$ number of iterations

b) $E' \subseteq E - A$

c) $|C| = 2\cdot|A|$

d) no two edges in A share an endpoint

e) edges in E', edges in A don't share endpoints

f) C covers E − E'

At termination, $\underbrace{|C^*| \ge |A|}$ (INV d → any cover must include one endpoint for each edge in A)

$\underbrace{|C| = 2\cdot|A|}$ (INV c)

$$\overbrace{|C| = 2\cdot|A|} \le 2\cdot|C^*|$$

METRIC-TSP

$w(u,v) \le w(u,x) + w(x,v)$

METRIC-TSP: Given fully connected, weighted G with weights that obey the triangle inequality, and a bound k, determine if G has a tour of total weight at most k

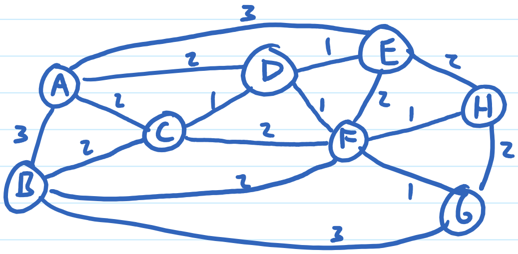
↑
NP-complete

doesn't obey triangle inequality: airfares $\underbrace{BWI-DFW}_{\$67} > \underbrace{BWI-MCO}_{\$18} + \underbrace{MCO-DFW}_{\$37}$

(if only I weren't under a stay-at-home order!)

does obey triangle inequality: Euclidean space (shortest distance is a straight line)
distances on sphere (great circle)

APPROX-METRIC-TSP(G) outputs a tour H of total weight $\le 2 \cdot |\overbrace{H^*}^{\text{tour of lowest total weight}}|$ (a 2-approximation)



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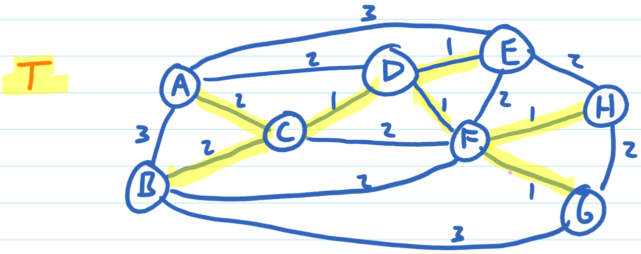
distances on sphere (great circle)

```
APPROX-METRIC-TSP(G)
T ← MST(G)
```

outputs a tour H of total weight $\le 2 \cdot |\overbrace{H^*}^{\text{tour of lowest total weight}}|$ (a 2-approximation)

(polynomial time via Prim's or Kruskal's algorithms)



METRIC-TSP

$w(u,v) \le w(u,x) + w(x,v)$

METRIC-TSP: Given fully connected, weighted G with weights that obey the triangle inequality, and a bound k, determine if G has a tour of total weight at most k

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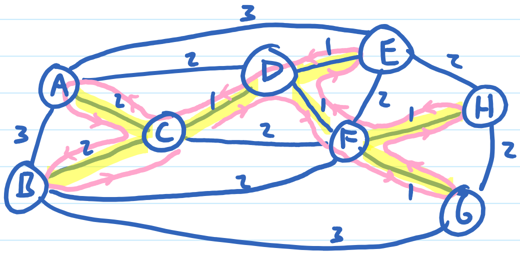
does obey triangle inequality: Euclidean space (shortest distance is a straight line)
distances on sphere (great circle)

tour of lowest total weight

APPROX-METRIC-TSP(G) outputs a tour H of total weight $\le 2 \cdot |H^*|$ (a 2-approximation)

$T \leftarrow MST(G)$ (polynomial time via Prim's or Kruskal's algorithms)



T

CDFGFHFDEDCACBC = S'

$w(S') \overset{\downarrow}{=} 2 \cdot w(T)$

back and forth along each edge

METRIC-TSP

$w(u,v) \le w(u,x) + w(x,v)$

METRIC-TSP: Given fully connected, weighted G with weights that obey the triangle inequality, and a bound k, determine if G has a tour of total weight at most k

↑ NP-complete

doesn't obey triangle inequality: airfares $\underbrace{\text{BWI-DFW}}_{\$67} > \underbrace{\text{BWI-MCO}}_{\$18} + \underbrace{\text{MCO-DFW}}_{\$37}$

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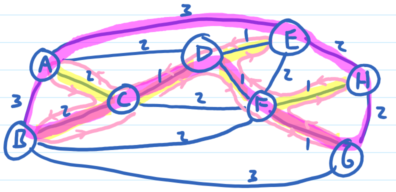
does obey triangle inequality: Euclidean space (shortest distance is a straight line)
distances on sphere (great circle)

tour of lowest total weight

```
APPROX-METRIC-TSP(G)
    T ← MST(G)
    v ← any vertex
    S ← preorder(T, v)
    return S + v
```

outputs a tour H of total weight $\le 2 \cdot |H^*|$ (a 2-approximation)

(polynomial time via Prim's or Kruskal's algorithms)



T

$CDFG\cancel{F}H\cancel{F}\cancel{D}ED\cancel{D}\cancel{C}A\cancel{C}BC = S'$

$CDFGHEABC = S$

triangle inequality ↓ optimal tour is spanning tree + extra edge ↓

$w(S) \le w(S') = 2 \cdot w(T) \le 2 \cdot w(H^*)$

↓ back and forth along each edge

Approximation for General TSP

TSP is not 2-approximable unless P=NP

Given undirected G, construct fully connected, undirected, weighted G′ by

1) copying the vertices of G
2) adding edges between all pairs of vertices
3) setting $w(u,v) = \begin{cases} 1 & \text{if } (u,v) \text{ exists in } G \\ 2+n & \text{otherwise} \end{cases}$

```
HAMILTONIAN-CYCLE(G)
  build G′ as above
  $H \leftarrow$ APPROX-TSP(G′)      (so $w(H) \le 2 \cdot w(H^*)$)
  if $|H| \le 2n$ then output YES
                else output NO
```

poly-time 2-approximation for TSP

↓

poly-time algorithm for HC

↓

P=NP

G has a HC → G′ has a tour H^* with $w(H^*)=n$ → APPROX-TSP returns an H with $w(H) \le 2 \cdot w(H^*) = 2n$

→ output is YES

G has no HC → every tour includes at least one edge with weight $2+n$

→ APPROX-TSP returns an tour H with $w(H) \ge (n-1)+(2+n) = 2n+1$

→ output is NO

Approximation for General TSP

TSP is not $\overset{k}{\boxed{2}}$-approximable unless P = NP

Given undirected G, construct fully connected, undirected, weighted G′ by

1) copying the vertices of G
2) adding edges between all pairs of vertices
3) setting $w(u,v) = \begin{cases} 1 & \text{if } (u,v) \text{ exists in } G \\ \underset{2+(k-1)n}{\boxed{2+n}} & \text{otherwise} \end{cases}$

```
HAMILTONIAN-CYCLE(G)
    build G′ as above
    H ← APPROX-TSP(G′)        (so $w(H) \le 2 \cdot w(H^*)$)
    if $|H| \le \underset{kn}{\boxed{2n}}$ then output YES
        else output NO
```

poly-time 2-approximation for TSP
↓
poly-time algorithm for HC
↓
P=NP

G has a HC → G′ has a tour H^* with $w(H^*) = n$ → APPROX-TSP returns an H with $w(H) \le \overset{k}{\boxed{2}}\, w(H^*) = \overset{kn}{\boxed{2n}}$
→ output is YES

G has no HC → every tour includes at least one edge with weight $2+n$
→ APPROX-TSP returns an tour H with $w(H) \ge (n-1) + \underset{2+(k-1)n}{\boxed{(2+n)}} = \overset{kn+1}{\boxed{2n+1}}$
→ output is NO