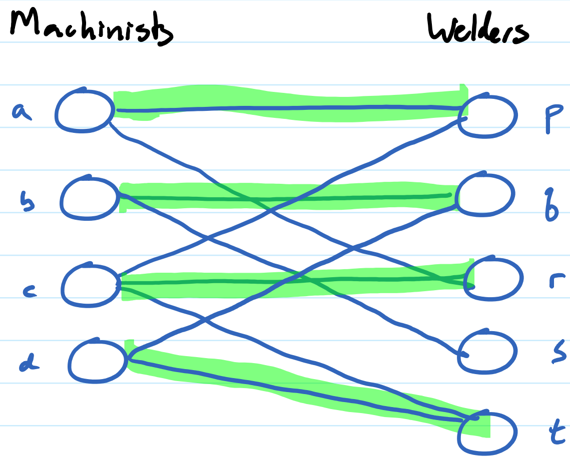
Bipartite Matching



a is willing to work with p and
p is willing to work with a

Maximum Bipartite Matching: Given a bipartite graph, find a matching of maximum size

Bipartite Matching

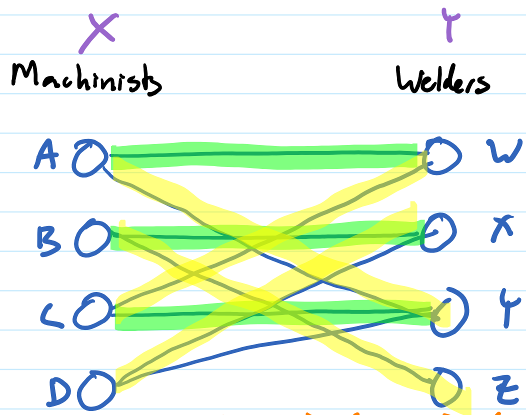
$X \cap Y = \emptyset$ $\quad X, Y \neq \emptyset$

$X \cup Y = V$

Bipartite Graph: V can be partitioned into X, Y s.t. all edges (u,v) have $u \in X$ and $v \in Y$ or $u \in Y$ and $v \in X$

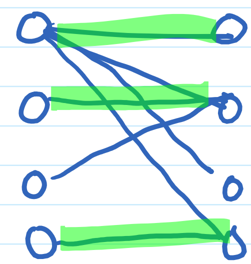
Matching in a bipartite graph: subset of edges M s.t.

$(x_1, y_1) \in M$
$(x_2, y_2) \in M$
$\downarrow$
$x_1 \neq x_2$ and $y_1 \neq y_2$
(or $x_1 = x_2$ and $y_1 = y_2$)



(A, Y) (C, W) (B, Z)
(D, X)

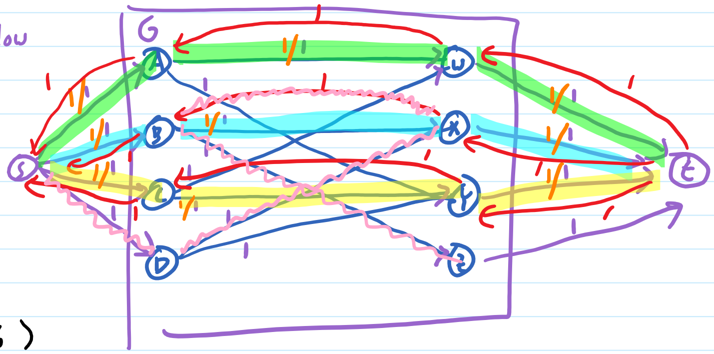
edge (m, w) means m will work with w and vice versa



Problem: find matching of maximum size

Bipartite Matching solved by Maximum Flow

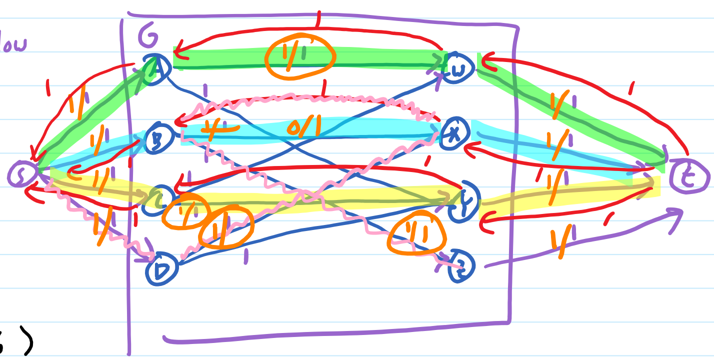
G′: input to max flow
needs
source
sink
capacity
direction



MAX-BIPARTITE-MATCH (G)

1) construct G′ as above
2) find max flow f of G′
3) output $M = \{ (x,y) \in G \text{ s.t. } f(x,y) = 1 \}$

lou



G)

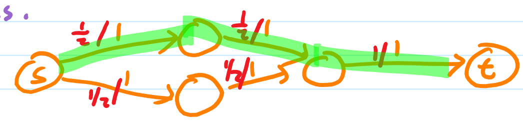
$M = \{ (A,u), (C,Y), (B,Z), (D,X) \}$

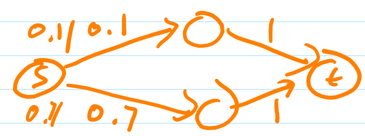
LEMMA 1: There is an integer-valued flow f in G′ with $v(f) = k$

$\Updownarrow$

There is a matching M in G with $|M| = k$

LEMMA 2: For directed graph G with integer capacities, then there is a max flow that has integer capacities.





THM: For bipartite G, max flow f in G′ gives max matching

Proof: Let f be max flow, M be corresponding matching

f is integer-valued — L2

$|M| = v(f)$ — L1 $\downarrow$

Suppose M not maximum: M′ is matching with $|M'| > |M|$

Then there is corresponding flow f′ with $v(f') = |M'| > |M| = v(f)$ — L1 $\uparrow$

So f is not max flow $\Rightarrow\Leftarrow$

$\therefore$ So M is maximum

DEF: s-t cut is

THM: Let f be a flow, (A,B) be an s-t cut.

Proof:
$$v(f) = f^{out}(s)$$
$$= f^{out}(s) - f^{in}(s)$$

$f^{out}(v) - f^{in}(v) = 0$ for all $v \in A - \{s\}$

$$v(f) = \sum_{v \in A} f^{out}(v) - f^{in}(v)$$

$$= \sum_{v \in A} \sum_{(v,x) \in E} f(v,x) - \sum_{(u,v) \in E} f(u,v)$$

$$= \sum_{v \in A} \sum_{\substack{(v,x) \in E \\ x \notin A}} f(v,x) - \sum_{\substack{(u,v) \in E \\ u \notin A}} f(u,v)$$

$$= f^{out}(A) - f^{in}(A)$$

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$\Updownarrow$

There is a matching M in G with $|M| = k$

Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}$

M is a matching in G

$(x,y) \in M \rightarrow (x,y) \in G$

can't have $(x,y_1), (x,y_2) \in M$, $y_1 \neq y_2$

can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$$v(f) = f^{out}(A)$$

$$= \sum_{\substack{(x,y) \in E' \\ x \in A \\ y \notin A}} f(x,y)$$

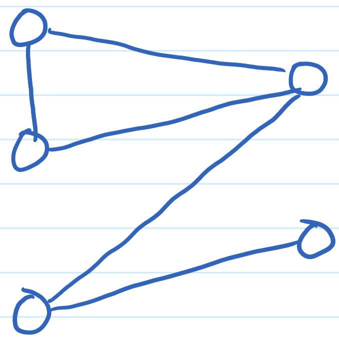
$$= \sum_{\substack{(x,y) \in E' \\ x \in X \\ y \in Y \\ f(x,y) = 1}} f(x,y)$$

$$= |\{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}|$$

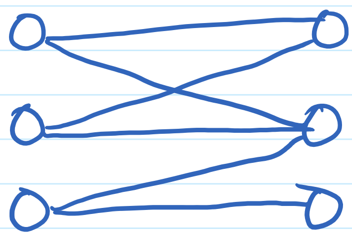
$$= |M|$$

Bipartite or Not?

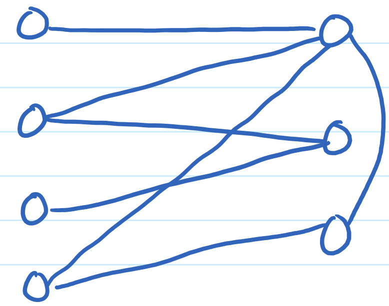
a)



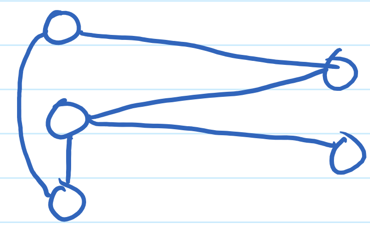
b)



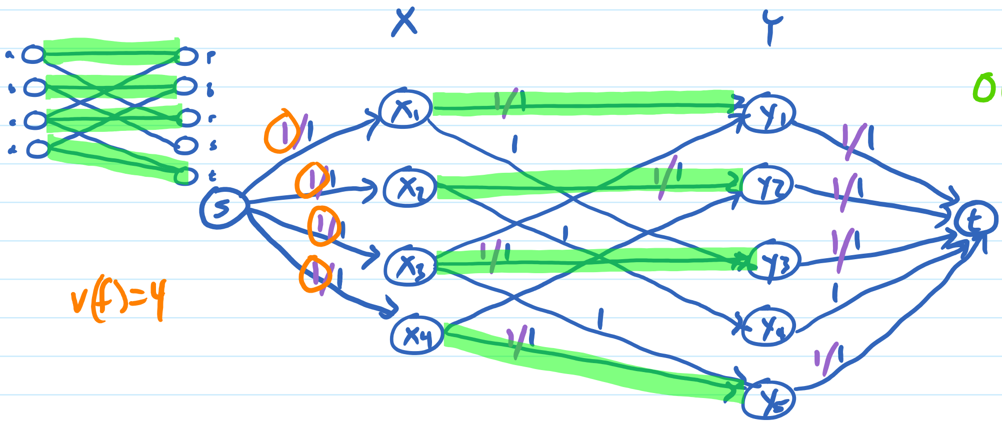
c)



d)



Bipartite Matching solved by Maximum Flow



bipartite graph G

$O(n+m)=O(m)$ ↓

capacity graph G′

$O(Cm)=O(nm)$ ↓ Ford-Fulkerson

maximum flow f

$O(m)$ ↓

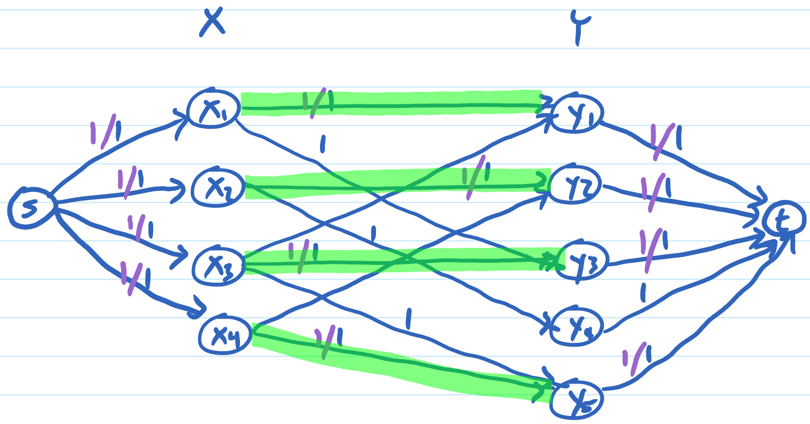
maximum matching M

total: $O(nm)$

G′ has same vertices as G, with source s and sink t added
and edges (s, x_i) for all $x_i \in X$, (x_i, y_j) for all $(x_i, y_j) \in E$, and (y_j, t) for all $y_j \in Y$
all edges have capacity 1

$$M = \{ (x_i, y_j) \text{ such that } f(x_i, y_j) = 1 \}$$

Bipartite Matching solved by Maximum Flow



bipartite graph G

↓

capacity graph G'

↓ Ford-Fulkerson

maximum flow f

↓

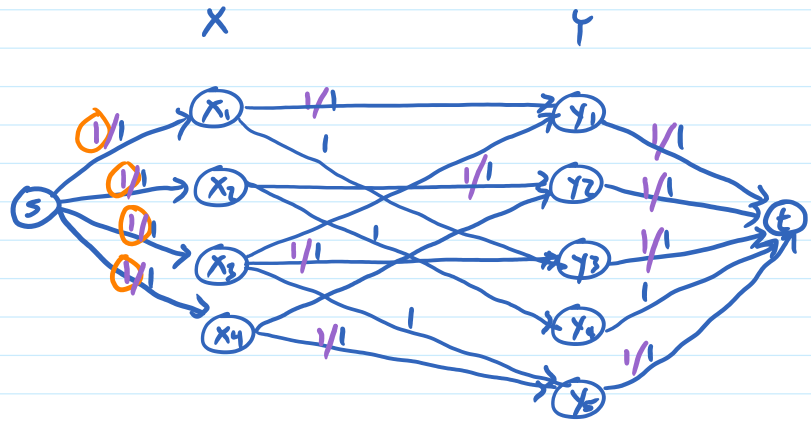
maximum matching M

G' has same vertices as G, with source s and sink t added and edges (s, x_i) for all $x_i \in X$, (x_i, y_j) for all $(x_i, y_j) \in E$, and (y_j, t) for all $y_j \in Y$

all edges have capacity 1

$$M = \{ (x_i, y_j) \text{ such that } f(x_i, y_j) = 1 \}$$

Bipartite Matching solved by Maximum Flow



v(f)=4

bipartite graph G
↓
capacity graph G′
↓ Ford-Fulkerson
maximum flow f
↓
maximum matching M

G′ has same vertices as G, with source s and sink t added
and edges (s, x_i) for all $x_i \in X$, (x_i, y_j) for all $(x_i, y_j) \in E$, and (y_j, t) for all $y_j \in Y$
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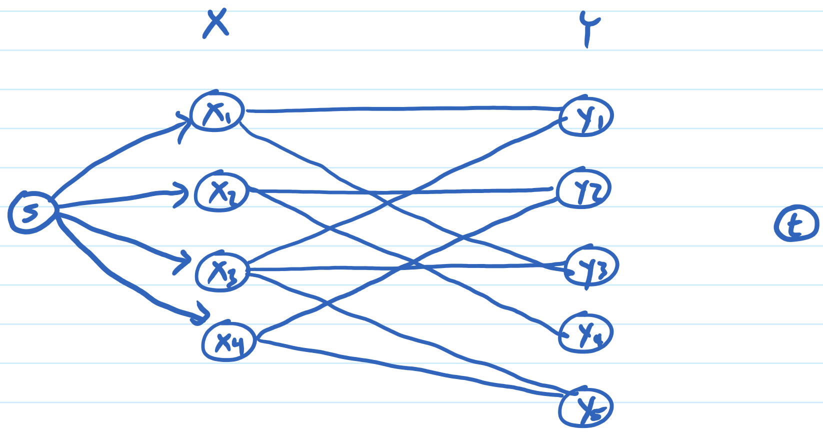
Bipartite Matching solved by Maximum Flow



bipartite graph G

$\downarrow$

capacity graph G'

$\downarrow$ Ford-Fulkerson

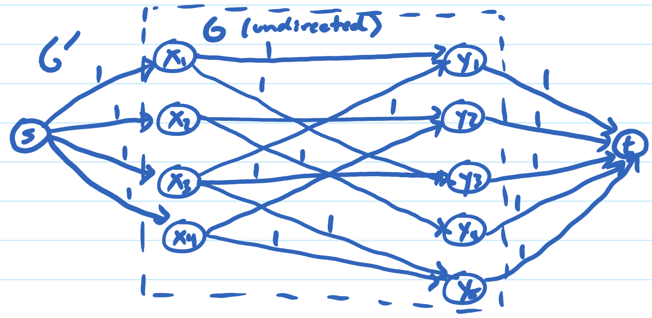
maximum flow f

$\downarrow$

maximum matching M

G' has same vertices as G, with source s and sink t added and edges (s, x_i) for all $x_i \in X$,

Bipartite Matching solved by Maximum Flow



LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$\Updownarrow$

There is a matching M in G with $|M| = k$

LEMMA 2: For directed graph G with integer capacities, then there is a max flow with integer values.

Proof: the flow returned from Ford-Fulkerson is integer-valued and maximum

THM: For bipartite G, max flow f in G' gives max matching

Proof: Let f be max flow

f is integer-valued LEMMA 2

Find M s.t. $|M| = k$ LEMMA 1 $\downarrow$ $(M = \{(x_i, y_j) \text{ s.t. } f(x_i, y_j) = 1\})$

Suppose M not maximum: M' is matching with $|M'| > |M|$

Then there is corresponding flow f' with $v(f') = |M'| > |M| = v(f)$ LEMMA 1 $\uparrow$

So f is not max flow

So M is maximum

DEF: s-t cut is partition of V into A, B s.t. $s \in A$, $t \in B$

THM: Let f be a flow, (A, B) be an s-t cut. Then $f^{out}(A) - f^{in}(A) = v(f)$

$f^{in}(B) - f^{out}(B)$

Proof:

$$v(f) = f^{out}(s)$$ def

$$= f^{out}(s) - f^{in}(s)$$ no edges in so $f^{in}(s) = 0$

$$f^{out}(v) - f^{in}(v) = 0 \text{ for all } v \in A - \{s\}$$ conservation

$$v(f) = \sum_{v \in A} f^{out}(v) - f^{in}(v)$$ $v \neq s$ terms are 0

$$= \sum_{v \in A}\left(\sum_{(v,x) \in E} f(v,x) - \sum_{(u,v) \in E} f(u,v)\right)$$ def f^{out}, f^{in}; $\sum_v \sum_{(v,u)} = \sum_{(v,u)}$

$$= \sum_{v \in A}\left(\sum_{\substack{(v,x) \in E \\ x \notin A}} f(v,x) - \sum_{\substack{(u,v) \in E \\ u \notin A}} f(u,v)\right)$$ edges $A \to A$ cancel

$$= f^{out}(A) - f^{in}(A)$$ rearrange; def $f^{in}(A)$, $f^{out}(A)$

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$$\Updownarrow$$

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Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}$

M is a matching in G

$(x,y) \in M \rightarrow (x,y) \in G$ — no edges added between X, Y

can't have $(x,y_1), (x,y_2) \in M$, $y_1 \neq y_2$ — otherwise $f(x,y_1) = f(x,y_2) = 1$, so $f^{out}(x) \geq 2$ so $f^{in}(x) \geq 2$ so $f(s,x) = 2$ $\rightarrow\leftarrow$

can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$ — similar

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$$v(f) = f^{out}(A) - f^{in}(A) = f^{out}(A)$$ prev THM; const. of G' allows no edges into A

$$= \sum_{\substack{(x,y) \in E' \\ x \in A \\ y \notin A}} f(x,y)$$ def

$$= \sum_{\substack{(x,y) \in E' \\ x \in X \\ y \in Y \\ f(x,y) = 1}} f(x,y)$$ all other flows are 0

$= \left| \{ (x,y) \mid x \in X, y \in Y, f(x,y) = 1 \} \right|$

$= |M|$ substitution

$\Leftarrow$ similar

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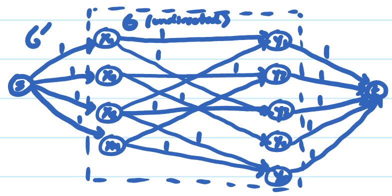

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$v(f') = |M'| > |M| = v(f)$ LEMMA 1 $\uparrow$

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$\therefore$ So M is maximum

Proof:

$v(f) = f^{out}(s)$ def

$= f^{out}(s) - f^{in}(s)$ no edges in so $f^{in}(s) = 0$

$f^{out}(v) - f^{in}(v) = 0$ for all $v \in A - \{s\}$ conservation

$v(f) = \sum_{v \in A} f^{out}(v) - f^{in}(v)$ $v \neq s$ terms are 0

$= \sum_{v \in A} \left(\sum_{(v,x) \in E} f(v,x) - \sum_{(u,v) \in E} f(u,v) \right)$ def f^{out}, f^{in}; $\sum_v \sum_{(v,x)} = \sum_{(v,x)}$

$= \sum \left(\sum f(v,x) - \sum f(u,v) \right)$ edges $A \to A$ cancel

$$= \sum_{v \in A} \left(\sum_{\substack{(v,x) \in E \\ x \notin A}} f(v,x) - \sum_{\substack{(u,v) \in E \\ u \notin A}} f(u,v) \right)$$ edges $A \to A$ cancel

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can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$ — similar

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$$v(f) = f^{out}(A) - f^{in}(A) = f^{out}(A)$$ prev THM; const. of G' allows no edges into A

$$= \sum_{\substack{(x,y) \in E' \\ x \in A \\ y \notin A}} f(x,y)$$ def

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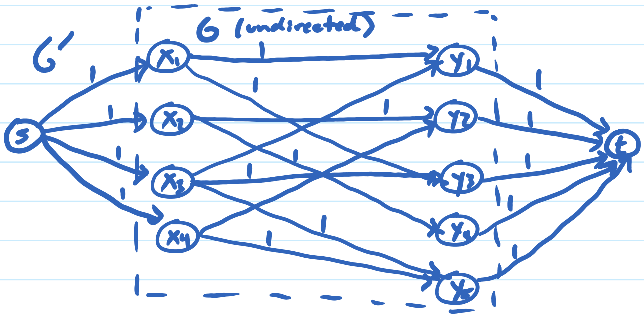
$$= \left| \{ (x,y) \mid x \in X, y \in Y, f(x,y) = 1 \} \right|$$

$$= |M|$$ substitution

$\Leftarrow$ similar

Bipartite Matching solved by Maximum Flow



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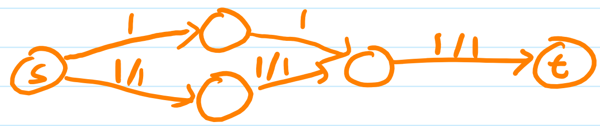

THM: Let f be a flow, (A,B) be an s-t cut. Then $\underset{\overset{\parallel}{f^{in}(B)}}{f^{out}(A)} - \underset{\overset{\parallel}{f^{out}(B)}}{f^{in}(A)} = v(f)$

Proof:

$$v(f) = f^{out}(s)$$ def

$$= f^{out}(s) - f^{in}(s)$$ no edges in so $f^{in}(s)=0$

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can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$ — similar

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$$= |M|$$ substitution

$\Leftarrow$ similar