Showing NP-Completeness

NP-complete: Problem X is NP-complete if 1) $X \in NP$
2) $Y \le_p X$ for all $Y \in NP$

To show X is NP-complete: 1) show $X \in NP$
2) show $Y \le_p X$ for some NP-complete Y

Let $Z \in NP$. Then $Z \le_p Y$ and so $Z \le_p X$

So, by transitivity, $Z \le_p X$ for all $Z \in NP$ and hence X is NP-complete

Showing NP-Completeness

NP-complete: Problem X is NP-complete if 1) $X \in NP$
2) $Y \le_p X$ for all $Y \in NP$

To show X is NP-complete: 1) show $X \in NP$
2) show $Y \le_p X$ for some NP-complete Y

Let $Z \in NP$. Then $Z \le_p Y$ and so $Z \le_p X$

So, by transitivity, $Z \le_p X$ for all $Z \in NP$ and hence X is NP-complete

Karp (many-one) reduction
$Y \le_m X$

<u>$Y(y)$</u>
build x from y
result $\leftarrow X(x)$
return result

<u>$VC(G)$</u>
$G' \leftarrow G$, $k' \leftarrow n-k$
result $\leftarrow IS(G', k')$
return result

<u>$HC(G)$</u>
$G' \leftarrow G$ with $w(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 2 & \text{otherwise} \end{cases}$, $k' \leftarrow n$
result $\leftarrow TSP(G', k')$
return result

Hamiltonian Cycle $\le_m$ Hamiltonian Path (and hence $HC \le_p HP$)

HP in NP

Hamiltonian Cycle: given undirected G, determine if G has a Hamiltonian cycle

Hamiltonian Path: given undirected G, determine if G has a Hamiltonian path

HP $\in$ NP:

HP-VERIFY(G, p)
- if every vertex in G appears exactly once in p — $O(n\underbrace{k}_{\text{length of } p})$
  - for consecutive v_i, v_{i+1} in p — $O(k)$ iterations
    - if (v_i, v_{i+1}) is not an edge — $O(n)$ per iteration
      - return NO — $O(nk)$ for the loop
  - return YES
- return NO — $O(nk)$ total

If G has Hamiltonian path $\underbrace{p}_{n \text{ vertices}}$, HP-VERIFY$(G,p)$ = YES

If G has no Hamiltonian path, HP-VERIFY(G,p) = NO for all p

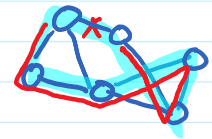
HC <= HP

$HC \le_m HP$:

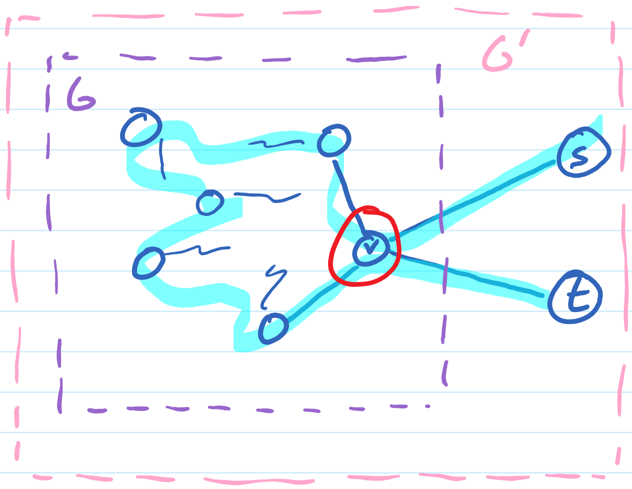
HC(G)
$G' \leftarrow$???
result $\leftarrow$ HP(G')
return result

want to create G' from G so that
G has HC
$\Updownarrow$
G' has HP

has HC $\rightarrow$ has HP



HP but no HC



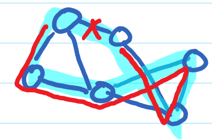
HC <= HP

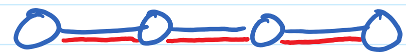
$HC \le_m HP$:

$\underline{HC(G)}$
$G' \leftarrow$???
result $\leftarrow HP(G')$
return result

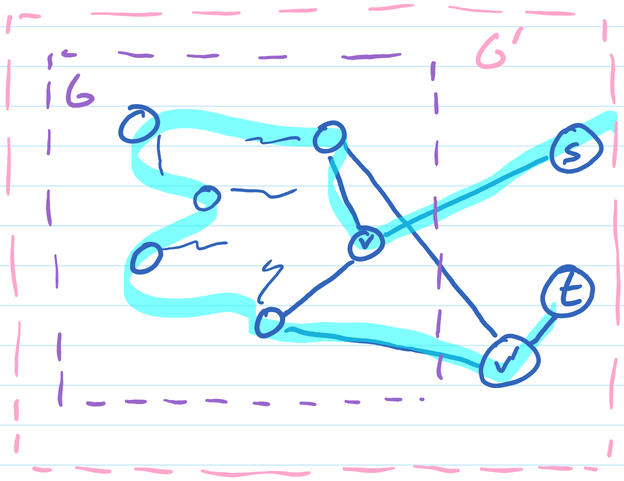
want to create G' from G so that
G has HC
$\Updownarrow$
G' has HP

has HC $\rightarrow$ has HP





HP but no HC



G' has HP $\rightarrow$ G has HC :

Let G have HP $s, \overbrace{v_1, \ldots, v_n}^{s,t,v' \text{ not in here, so all vertices, edges in } G}, t$

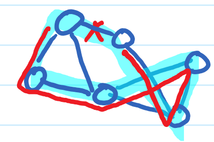
$v_1 = v$, $v_n = v'$

Then $v, v_1, \ldots, \underbrace{v_{n-1}, v}$ is HC in G

(v_{n-1}, v') is edge in G'
so (v_{n-1}, v) is edge in G
since v' is a copy of v

HC <= HP

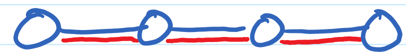
has HC → has HP



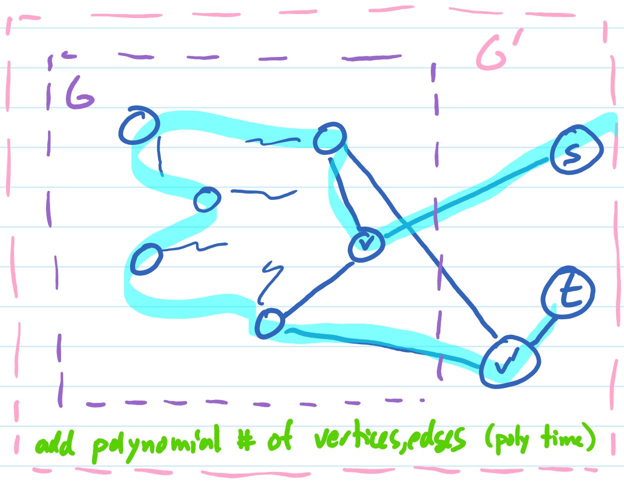
HC $\le_m$ HP :

HC(G)
G' ← ???
result ← HP(G')
return result

want to create G' from G so that
G has HC
↕
G' has HP



HP but no HC



G' has HP → G has HC :

s, t, v' not in here, so all vertices, edges in G

Let G have HP $s, \overbrace{v_1, \ldots\ldots, v_n}, t$
$v_1 = v$, $v_n = v'$

Then $v, v_1, \ldots, \underbrace{v_{n-1}, v}$ is HC in G

(v_{n-1}, v') is edge in G'
so (v_{n-1}, v) is edge in G
since v' is a copy of v

G has HC → G' has HP :
Let $v, v_1, \ldots, v_{n-1}, v$ be HC in G
Then $s, v, v_1, \ldots, v_{n-1}, v', t$ is an HP in G'

HP is NP-complete

HC is NP-complete

$HP \in NP$

$HC \leq_p HP$

$\therefore$ HP is NP-complete