Invariant

a) $\forall m,\ m \notin FreeM \longleftrightarrow \exists w$ s.t. $(m,w) \in Tent$
$\forall w,\ w \notin FreeW \longleftrightarrow \exists m$ s.t. $(m,w) \in Tent$
FreeM, FreeW keep track of unmatched M, W

b) $\forall w,\ w \in FreeW \longleftrightarrow \neg \exists m$ s.t. $(m,w) \in Invites$
free welders are exactly those with no received invitations

c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

d) $|Invites| = k$

e) $\forall w, j < k$, $MatchW_j(w) \neq NIL \rightarrow MatchW_{j+1}(w), \ldots, MatchW_k(w) \; != NIL$
The value of MatchW(w) after j iterations
once w receives first invitation, w is never free again

f) $\forall w$, $MatchW(w) = \max_m{}_w \; (m,w) \in Invites$
→ max over w's preference list
the m s.t. $(m,w) \in Tent$ (or NIL if no such m)
(or NIL if no such m) w is matched with most preferred machinist who has sent an invitation to w

g) $\forall m, w, w'$ if $(m,w) \in Invites$ and m prefers w' to w then $(m,w') \in Invites$
machinists send invitations in order of preference

h) $Tent \subseteq Invites$

Termination: when $k = n^2$, $|Invites| = n^2$ — INV d
and $Invites = M \times W$
so $\forall w \; \exists m$ s.t. $(m,w) \in Invites$
and $\forall w,\ w \notin FreeW$ — INV b
$\therefore \forall m,\ m \notin FreeM$ — INV c

Postcondition: Tent is a perfect matching
Tent is stable — INV c

Maintenance (hard part)

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

Suppose INV is T before loop and $\exists m \in FreeM, w$ s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish
↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t.
$\hat{m} \in Tent$ and $\hat{w} \in Tent$ and
$\hat{m}$ prefers $\hat{w}$ to $MatchM(\hat{m})$ and
$\hat{w}$ prefers $\hat{m}$ to $MatchW(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$
or
$MatchM(\hat{m}) \neq NIL$

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

4 cases:

Maintenance (hard part)

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

Suppose INV is T before loop and $\exists m \in FreeM, w$ s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t. $\hat{m} \in Tent$ and $\hat{w} \in Tent$ and $\hat{m}$ prefers $\hat{w}$ to $MatchM(\hat{m})$ and $\hat{w}$ prefers $\hat{m}$ to $MatchW(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$ ← or $MatchM(\hat{m}) \neq NIL$

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

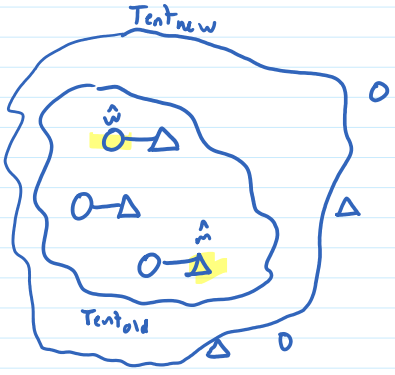
3 cases: i) $Match M_{new}(\hat{m}) = Match M_{old}(\hat{m})$ and $Match W_{new}(\hat{w}) = Match W_{old}(\hat{w})$

so $\hat{m},\hat{w} \in Tent_{old}$

$\hat{m}$ prefers $\hat{w}$ to $Match M_{old}(\hat{m}) = Match M_{new}(\hat{m})$

$\hat{w}$ prefers $\hat{m}$ to $Match W_{old}(\hat{w}) = Match W_{new}(\hat{w})$

$(\hat{m},\hat{w})$ is an instability wrt $Tent_{old}$ ⇒⇐



Maintenance (hard part)

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

Suppose INV is T before loop and $\exists m \in FreeM, w$ s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t. $\hat{m} \in Tent$ and $\hat{w} \in Tent$ and $\hat{m}$ prefers $\hat{w}$ to $MatchM(\hat{m})$ and $\hat{w}$ prefers $\hat{m}$ to $MatchW(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$ ← or $MatchM(\hat{m}) \neq NIL$

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

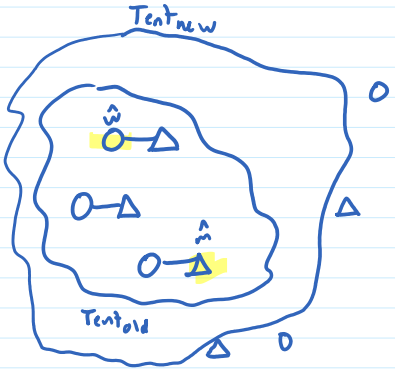
3 cases: i) $MatchM_{new}(\hat{m}) = MatchM_{old}(\hat{m})$ and $MatchW_{new}(\hat{w}) = MatchW_{old}(\hat{w})$

so $\hat{m},\hat{w} \in Tent_{old}$

$\hat{m}$ prefers $\hat{w}$ to $MatchM_{new}(\hat{m}) = MatchM_{old}(\hat{m})$

$\hat{w}$ prefers $\hat{m}$ to $MatchW_{new}(\hat{w}) = MatchW_{old}(\hat{w})$ (def instability)

$(\hat{m},\hat{w})$ is an instability wrt $Tent_{old}$ def ⇒⇐



Maintenance (hard part)

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

Suppose INV is T before loop and $\exists m \in FreeM, w$ s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t.
$\hat{m} \in Tent$ and $\hat{w} \in Tent$ and
$\hat{m}$ prefers $\hat{w}$ to $MatchM(\hat{m})$ and
$\hat{w}$ prefers $\hat{m}$ to $MatchW(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$
or
$MatchM(\hat{m}) \neq NIL$

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

3 cases: i) $MatchM_{new}(\hat{m}) \neq MatchM_{old}(\hat{m})$

so $\hat{w} \in Tent_{old}$ and $MatchW_{old}(\hat{w}) = MatchW_{new}(\hat{w}) = \hat{\hat{m}}$

m prefers $\hat{w}$ to w
$\hat{w}$ prefers m to $\hat{\hat{m}}$ — def instability

$(m,\hat{w}) \in Invites$ — choice of w

$\hat{w}$ prefers $\hat{\hat{m}}$ to m ⇒⇐ INV f

Maintenance (hard part)

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

Suppose INV is T before loop and $\exists m \in FreeM, w$ s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t. $\hat{m} \in Tent$ and $\hat{w} \in Tent$ and $\hat{m}$ prefers $\hat{w}$ to $MatchM(\hat{m})$ and $\hat{w}$ prefers $\hat{m}$ to $MatchW(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$ or $MatchM(\hat{m}) \neq NIL$

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

3 cases: i) $MatchM_{new}(\hat{m}) \neq MatchM_{old}(\hat{m})$

so $\hat{w} \in Tent_{old}$ and $MatchW_{old}(\hat{w}) = MatchW_{new}(\hat{w}) = \hat{\hat{m}}$

$\hat{m}$ prefers $\hat{w}$ to w

$\hat{w}$ prefers m to $\hat{\hat{m}}$ def instability

$(m,\hat{w}) \in Invites$ choice of w

$\hat{w}$ prefers $\hat{\hat{m}}$ to m ⇒⇐ INV f

Maintenance (hard part)

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    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

Suppose INV is T before loop and $\exists m \in FreeM$, w s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t.
$\hat{m} \in Tent$ and $\hat{w} \in Tent$ and
$\hat{m}$ prefers $\hat{w}$ to $Match M(\hat{m})$ and
$\hat{w}$ prefers $\hat{m}$ to $Match W(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$ ← or $Match M(\hat{m}) \neq NIL$

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

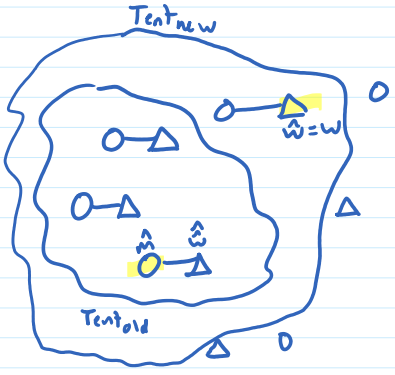
3 cases: iii) $Match M_{new}(\hat{m}) = Match M_{old}(\hat{m})$ and
$Match W_{new}(\hat{w}) \neq Match W_{old}(\hat{w})$

so $\hat{w} = w$

find $\hat{\hat{w}}$ s.t. $(\hat{m},\hat{\hat{w}}) \in Tent_{old}$

$\hat{m}$ prefers w to $\hat{\hat{w}}$ def instability

$(\hat{m},\hat{\hat{w}}) \in Invites_{old}$ INV h
$(\hat{m}, w) \in Invites_{old}$ INV g
$w \notin FreeW_{old}$ INV b



Maintenance (hard part)

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Tentative <- {}
k <- 0

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    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

Suppose INV is T before loop and $\exists m \in FreeM$, w s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t.

$\hat{m} \in Tent$ and $\hat{w} \in Tent$ and $\hat{m}$ prefers $\hat{w}$ to $Match M(\hat{m})$ and $\hat{w}$ prefers $\hat{m}$ to $Match W(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$ ← or $Match M(\hat{m}) \neq NIL$

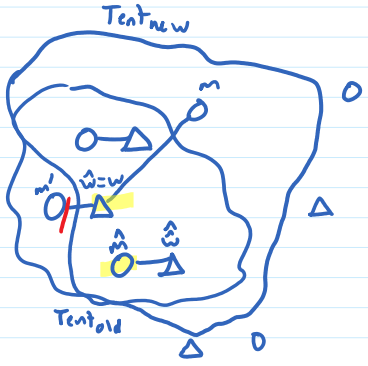
no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

3 cases: iii) $Match M_{new}(\hat{m}) = Match M_{old}(\hat{m})$ and $Match W_{new}(\hat{w}) \neq Match W_{old}(\hat{w})$

so $\hat{w} = w$

find $\hat{\hat{w}}$ s.t. $(\hat{m},\hat{\hat{w}}) \in Tent_{old}$



$\hat{m}$ prefers w to $\hat{\hat{w}}$ def instability

$(\hat{m},\hat{\hat{w}}) \in Invites_{old}$ INV h

$(\hat{m}, w) \in Invites_{old}$ INV g

$w \notin FreeW_{old}$ INV b

w prefers m to m' to $\hat{m}$

w prefers $\hat{m}$ to m ⇒⇐ def instability

Maintenance (hard part)

Suppose INV is T before loop and $\exists m \in FreeM$, w s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t.
$\hat{m} \in Tent$ and $\hat{w} \in Tent$ and
$\hat{m}$ prefers $\hat{w}$ to $MatchM(\hat{m})$ and
$\hat{w}$ prefers $\hat{m}$ to $MatchW(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$
or
$MatchM(\hat{m}) \neq NIL$

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

4 cases: iii) $MatchM_{new}(\hat{m}) = MatchM_{old}(\hat{m})$ and
$MatchW_{new}(\hat{w}) \neq MatchW_{old}(\hat{w})$

Maintenance (hard part)

Suppose INV is T before loop and $\exists m \in FreeM$, w s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t. $\hat{m} \in Tent$ and $\hat{w} \in Tent$ and $\hat{m}$ prefers $\hat{w}$ to $MatchM(\hat{m})$ and $\hat{w}$ prefers $\hat{m}$ to $MatchW(\hat{w})$

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$ ← or $MatchM(\hat{m}) \neq NIL$

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FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
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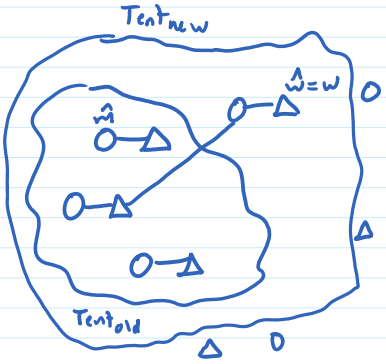
no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

4 cases: iii) $MatchM_{new}(\hat{m}) = MatchM_{old}(\hat{m})$ and $MatchW_{new}(\hat{w}) \neq MatchW_{old}(\hat{w})$



Maintenance (hard part)

Suppose INV is T before loop and $\exists m \in FreeM$, w s.t. $(m,w) \notin Invites$

c) Tent is a matching and stablish

↳ there is no $(\hat{m},\hat{w}) \notin Tent$ s.t. $\hat{m} \in Tent$ and $\hat{w} \in Tent$ and $\hat{m}$ prefers $\hat{w}$ to MatchM($\hat{m}$) and $\hat{w}$ prefers $\hat{m}$ to MatchW($\hat{w}$)

$\exists \hat{w}$ s.t. $(\hat{m},\hat{w}) \in Tent$ ← or MatchM($\hat{m}$) ≠ NIL

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}
k <- 0

while there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
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        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
    k <- k+1
return Tentative
```

no instabilities wrt $Tent_{old}$ INV c

Suppose there is an instability $(\hat{m},\hat{w})$ wrt $Tent_{new}$

Invariant

b) $\forall w,\ w \in FreeW \longleftrightarrow \neg \exists m$ s.t. $(m,w) \in Invites$] free welders are exactly those with no received invitations

c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

d) $|Invites| = k$

Termination: when $k = n^2$, $|Invites| = n^2$ — INV d
and $Invites = M \times W$
so $\forall w\ \exists m$ s.t. $(m,w) \in Invites$
and $\forall w,\ w \notin FreeW$ — INV b
$\therefore \forall m,\ m \notin FreeM$ — INV c

Postcondition: Tent is a perfect matching
Tent is stable — INV c