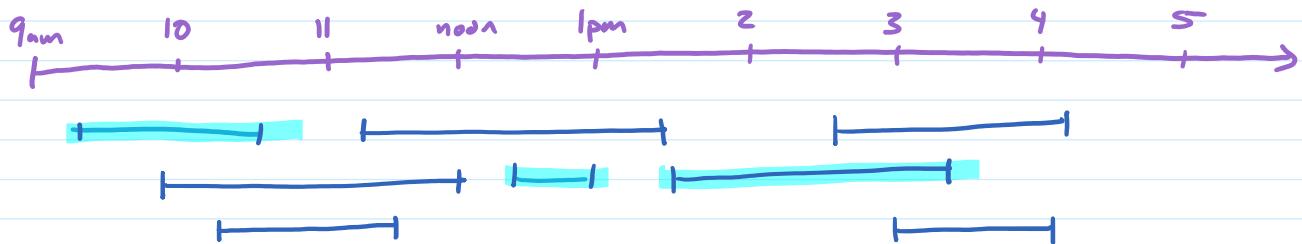


Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[i] < f[1], \dots, s[n] < f[n]$

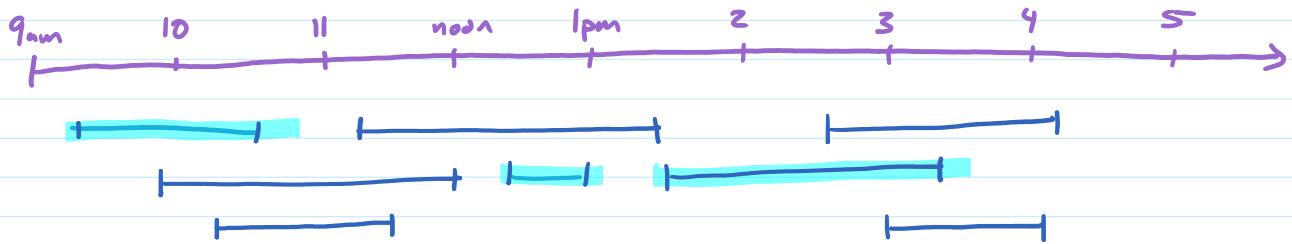
POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```
A ← []
k ← 0
R ← {1, ..., n}
while R ≠ ∅
    choose  $i \in R$  to minimize  $f[i]$ 
    append  $i$  to  $A$ 
     $R \leftarrow R - \{x \in R \mid s[x] < f[i]\}$ 
```

inv:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

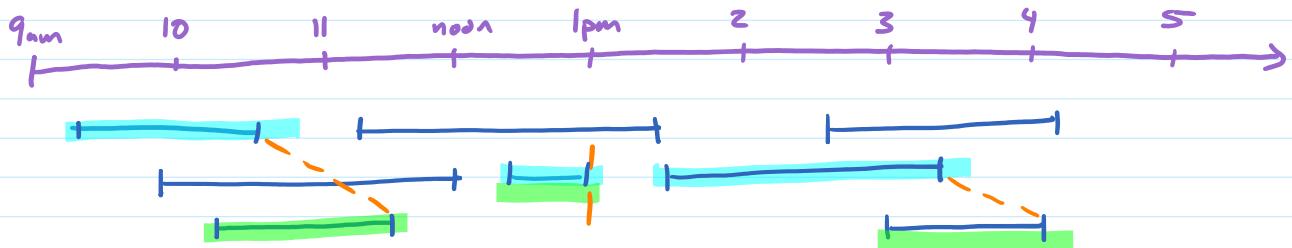
 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
    choose  $i \in R$  to minimize  $f[i]$ 
    append  $i$  to  $A$ 
     $k \leftarrow k + 1$ 
    remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

```

INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of finish  $j_1, \dots, j_m$   
 $f[a_s] \leq f[j_e]$  for  $1 \leq s \leq m$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

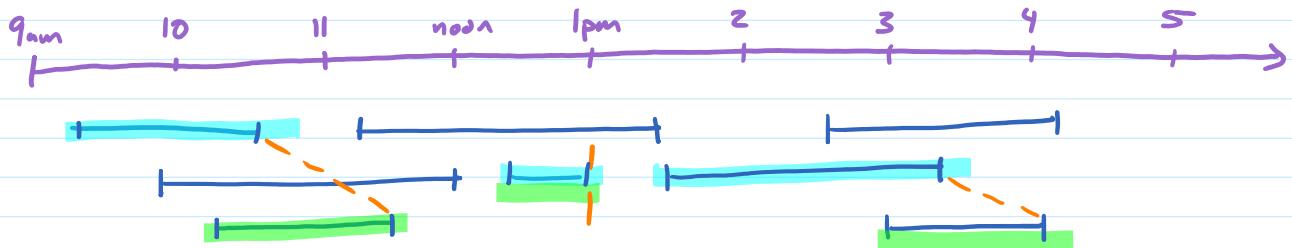
 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
    choose  $i \in R$  to minimize  $f[i]$ 
    append  $i$  to  $A$ 
     $k \leftarrow k + 1$ 
    remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

```

INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of finish  $j_1, \dots, j_m$   
 $f[a_1] \leq f[j_1] \leq \dots \leq f[a_k] \leq f[j_k]$  for  $1 \leq k \leq k$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

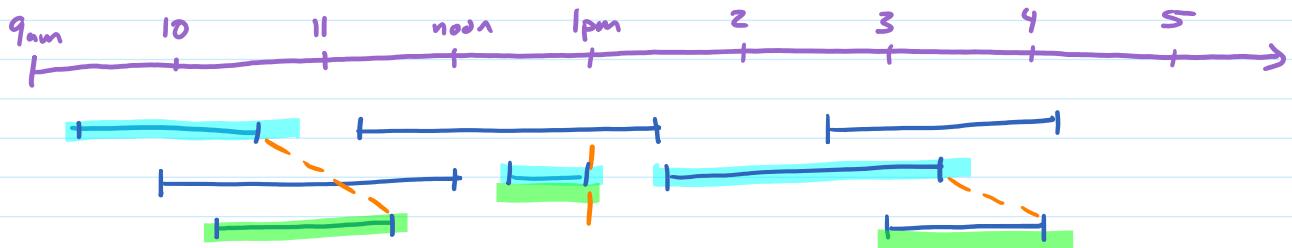
 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
    choose  $i \in R$  to minimize  $f[i]$ 
    append  $i$  to  $A$ 
     $k \leftarrow k + 1$ 
    remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

```

INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of finish  $j_1, \dots, j_m$   
 $f[a_s] \leq f[j_s]$  for  $1 \leq s \leq k$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
    choose  $i \in R$  to minimize  $f[i]$ 
    append  $i$  to  $A$ 
     $k \leftarrow k + 1$ 
    remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

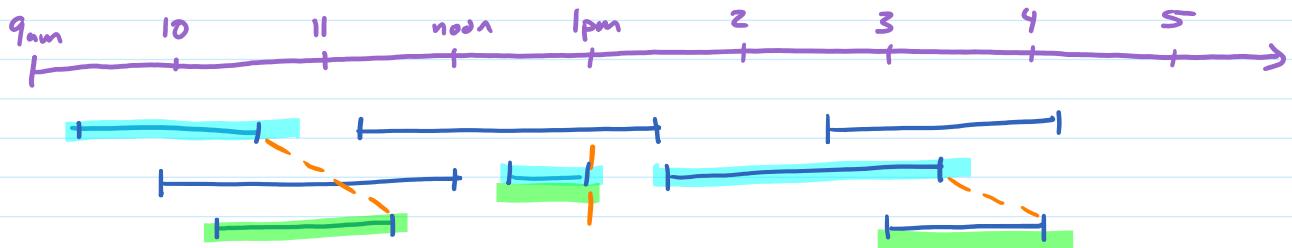
```

INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of finish  $j_1, \dots, j_m$   
 $f[a_1] \leq f[j_l]$  for  $1 \leq l \leq k$

Maintenance:

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
  choose  $i \in R$  to minimize  $f[i]$ 
  append  $i$  to  $A$ 
   $R \leftarrow R - \{x \in R \mid s[x] \geq f[i]\}$ 
   $k \leftarrow k + 1$ 

```

INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of  $\ell$  finish times  $j_1, \dots, j_m$   
 $f[a_\ell] \leq f[j_\ell]$  for  $1 \leq \ell \leq k$

Maintenance: e) [show  $R_{\text{new}} = \{x \in R_{\text{old}} \mid x \text{ is compatible with } A_{\text{new}}\}$ ]

$\subseteq$ : Let  $x \in R_{\text{new}}$ .

Then  $x \in R_{\text{old}}$

$R_{\text{new}} \subseteq R_{\text{old}}$

so  $x$  is compatible with  $A_{\text{old}}$   
 Also,  $s[x] \geq f[i]$

INV e

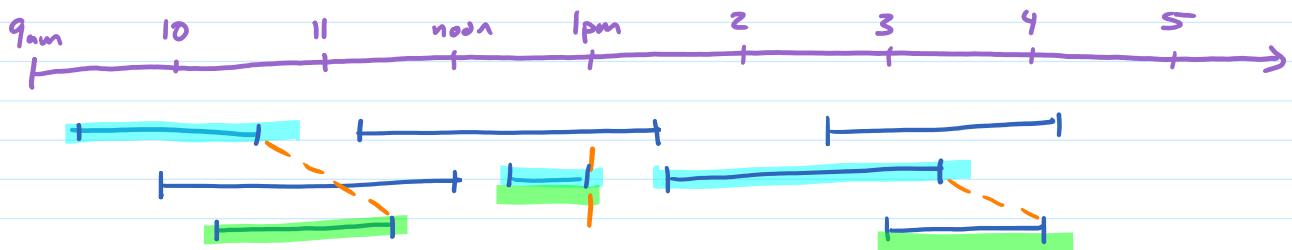
So  $x$  is compatible with  $i$   
 $\therefore x$  is compatible with  $A_{\text{new}}$

choice of  $R_{\text{new}}$

$A_{\text{new}} = A_{\text{old}} + i$

$s[i] \xrightarrow{i} f[i] \xrightarrow{x} s[x] \dots$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
  choose  $i \in R$  to minimize  $f[i]$ 
  append  $i$  to  $A$ 
   $k \leftarrow k + 1$ 
  remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

```

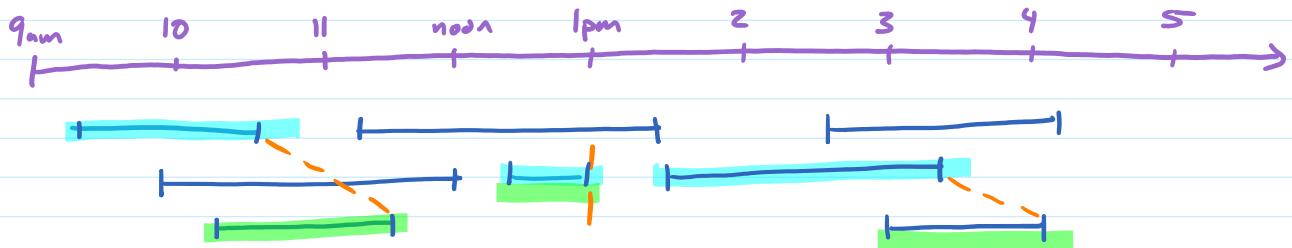
INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of  $\uparrow$  finish  $j_1, \dots, j_m$   
 $f[a_1] \leq f[j_1] \leq \dots \leq f[a_m] \leq f[j_m]$  for  $1 \leq l \leq k$

Maintenance: e) [show  $R_{\text{new}} = \{x \in R_{\text{old}} \mid x \text{ is compatible with } A_{\text{new}}\}$ ]

$\subseteq$ : Let  $x \in R_{\text{new}}$ .  
 Then  $x \in R_{\text{old}}$        $R_{\text{new}} \subseteq R_{\text{old}}$   
 so  $x$  is compatible with  $A_{\text{old}}$       INV e  
 Also,  $s[x] \geq f[i]$       choice of  $R_{\text{new}}$   
 So  $x$  is compatible with  $i$   
 $\therefore x$  is compatible with  $A_{\text{new}}$        $A_{\text{new}} = A_{\text{old}} + i$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

$A \leftarrow []$

$k \leftarrow 0$

$R \leftarrow \{1, \dots, n\}$

while  $R \neq \emptyset$

choose  $i \in R$  to minimize  $f[i]$

append  $i$  to  $A$

$k \leftarrow k + 1$

remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$

INV: a)  $|A| = k$

b) intervals in  $A$  are pairwise compatible

c)  $A$  is in order of increasing finish

d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$

e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$

f) for any optimal  $\Theta$  written in order  
of  $T$  finish  $j_1, \dots, j_m$

$f[a_s] \leq f[j_s]$  for  $1 \leq s \leq m$

Maintenance: e) [show  $R_{\text{new}} = \{x \in R \mid x \text{ is compatible with } A_{\text{new}}\}$ ]

$\geq$ : Suppose  $x$  is compatible with  $A_{\text{new}}$

Then  $x$  is compatible with  $A_{\text{old}}$

So  $x \in R_{\text{old}}$

Hence  $f[x] \geq f[:]$

and  $s[x] \geq f[:]$

$\therefore x \in R_{\text{new}}$



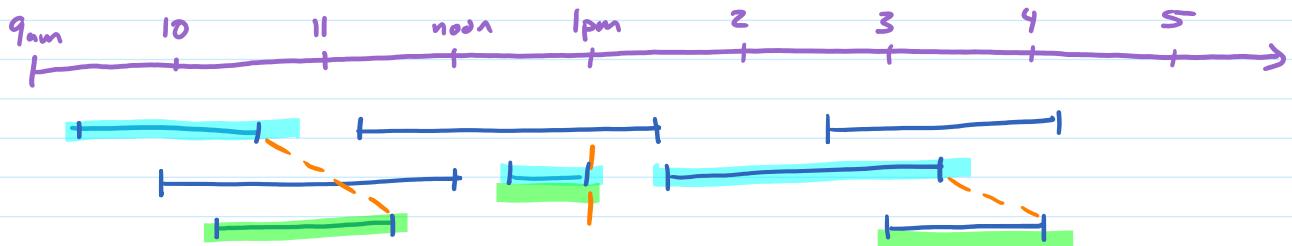
$A_{\text{old}} \subseteq A_{\text{new}}$

INV e

choice of  $i$

choice of  $R_{\text{new}}$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

$A \leftarrow []$

$k \leftarrow 0$

$R \leftarrow \{1, \dots, n\}$

while  $R \neq \emptyset$

choose  $i \in R$  to minimize  $f[i]$

append  $i$  to  $A$

$k \leftarrow k + 1$

remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$

INV: a)  $|A| = k$

b) intervals in  $A$  are pairwise compatible

c)  $A$  is in order of increasing finish

d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$

e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$

f) for any optimal  $\Theta$  written in order  
of  $\uparrow$  finish  $j_1, \dots, j_m$

$f[a_s] \leq f[j_s]$  for  $1 \leq s \leq k$

Maintenance: e) [show  $R_{\text{new}} = \{x \in R \mid x \text{ is compatible with } A_{\text{new}}\}$ ]

$\geq$ : Suppose  $x$  is compatible with  $A_{\text{new}}$

Then  $x$  is compatible with  $A_{\text{old}}$

So  $x \in R_{\text{old}}$

Hence  $f[x] \geq f[i]$

and  $s[x] \geq f[i]$

ii)  $x \in R_{\text{new}}$



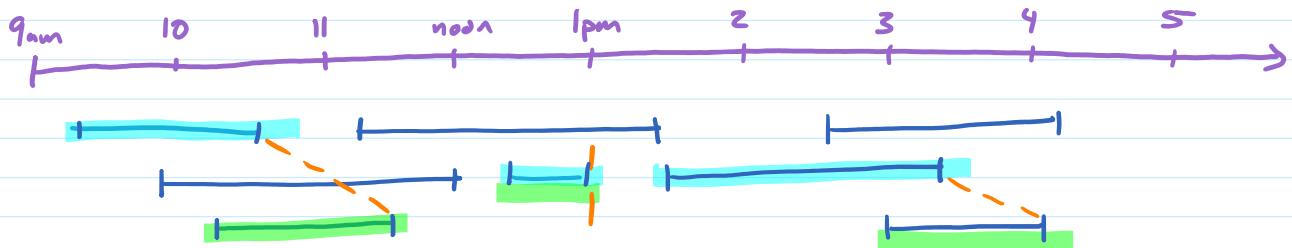
$A_{\text{old}} \subseteq A_{\text{new}}$

INV e

choice of  $i$

choice of  $R_{\text{new}}$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
  choose  $i \in R$  to minimize  $f[i]$ 
  append  $i$  to  $A$ 
   $k \leftarrow k + 1$ 
  remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

```

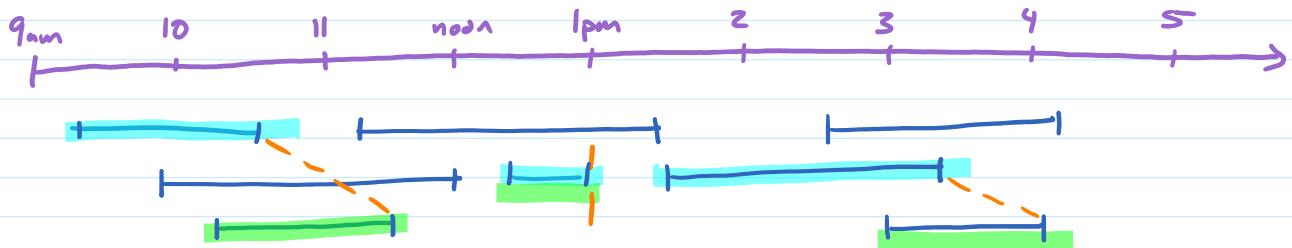
INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid X \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of  $\uparrow$  finish  $j_1, \dots, j_m$   
 $f[a_i] \leq f[j_l]$  for  $1 \leq i \leq k$

Maintenance: f) 2 cases: i)  $k=0$  then  $A = []$  and  $R = R_{\text{ini}}$  INV a and e  
and so  $f[a_i] \leq f[j_1]$  choice of  $i$

ii)  $k > 0$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[1] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
  choose  $i \in R$  to minimize  $f[i]$ 
  append  $i$  to  $A$ 
   $k \leftarrow k + 1$ 
  remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

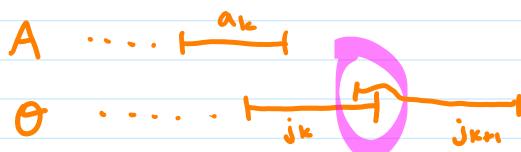
```

INV:

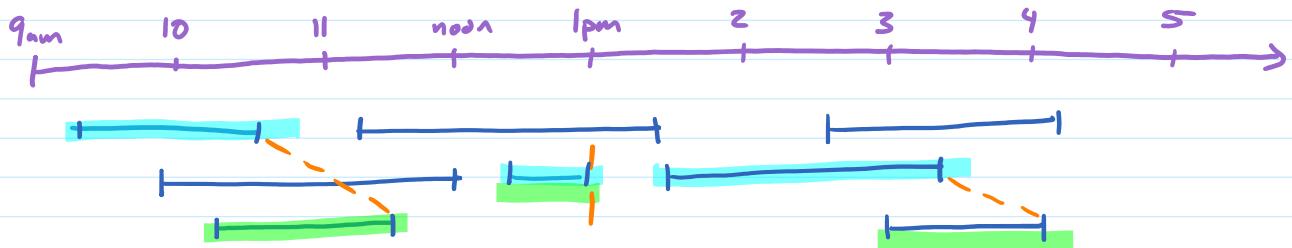
- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of finish  $j_1, \dots, j_m$   
 $f[a_i] \leq f[j_l]$  for  $1 \leq l \leq k$

Maintenance: f) 2 cases: i)  $k=0$  then  $A=[ ]$  and  $R=R_{in}$  INV a and e  
 and so  $f[a_i] \leq f[j_1]$  choice of  $i$

ii)  $k>0$  then  $f[a_k] \leq f[j_k]$   
 Also  $f[j_{k+1}] > f[j_k]$  INV f  
 $\Theta$  is sorted



Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[i] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

```

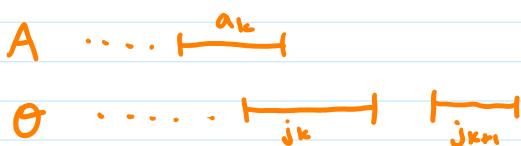
 $A \leftarrow []$ 
 $k \leftarrow 0$ 
 $R \leftarrow \{1, \dots, n\}$ 
while  $R \neq \emptyset$ 
  choose  $i \in R$  to minimize  $f[i]$ 
  append  $i$  to  $A$ 
   $k \leftarrow k + 1$ 
  remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$ 

```

INV:

- a)  $|A| = k$
- b) intervals in  $A$  are pairwise compatible
- c)  $A$  is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of finish  $j_1, \dots, j_m$   
 $f[a_i] \leq f[j_l]$  for  $1 \leq l \leq k$

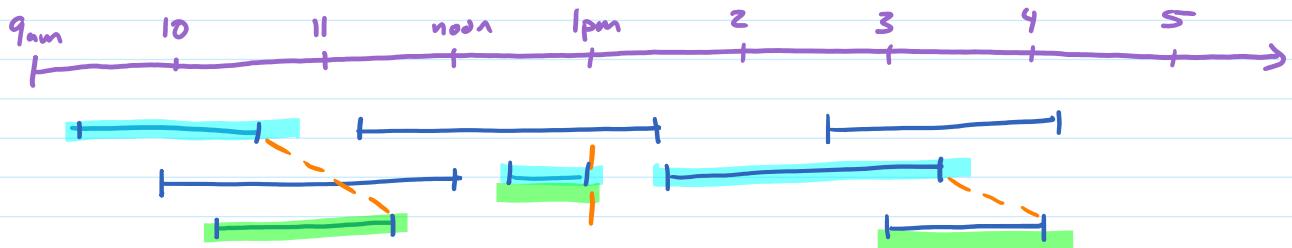
Maintenance: f) 2 cases : i)  $k=0$  then  $A = []$  and  $R = R_{in}$  INV a and e  
and so  $f[a_i] \leq f[j_1]$  choice of  $i$



ii)  $k > 0$  then  $f[a_k] \leq f[j_k]$   
Also  $f[j_{k+1}] > f[j_k]$   
and  $s[j_{k+1}] \geq f[j_k]$

INV f  
 $\Theta$  is sorted  
 $j_k, j_{k+1}$  compatible

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[i] < f[1], \dots, s[n] < f[n]$

POST: A is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

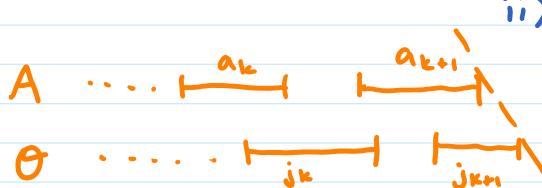
```

A ← []
k ← 0
R ← {1, ..., n}
while R ≠ ∅
    choose  $i \in R$  to minimize  $f[i]$ 
    append  $i$  to A
    k ← k + 1
    remove from R the intervals  $x$  with  $s[x] < f[i]$ 
  
```

INV:

- a)  $|A| = k$
- b) intervals in A are pairwise compatible
- c) A is in order of increasing finish
- d)  $R = \{x \in R \mid s[x] \geq \max_{y \in A} f[y]\}$
- e)  $R = \{x \in R \mid x \text{ is compatible w/ } A\}$
- f) for any optimal  $\Theta$  written in order of  $f[a_1] \leq f[j_1] \leq \dots \leq f[a_k] \leq f[j_k]$  for  $1 \leq l \leq k$

Maintenance: f) 2 cases: i)  $k=0$  then  $A = []$  and  $R = R_{\text{in}}$  INV a and e  
and so  $f[a_i] \leq f[j_i]$  choice of  $i$



ii)  $k > 0$

then  $f[a_k] \leq f[j_k]$

Also  $f[j_{k+1}] > f[j_k]$

and  $s[j_{k+1}] \geq f[j_k]$

$s[j_{k+1}] \geq f[a_k]$

$\therefore f[a_{k+1}] \leq f[j_{k+1}]$

INV f

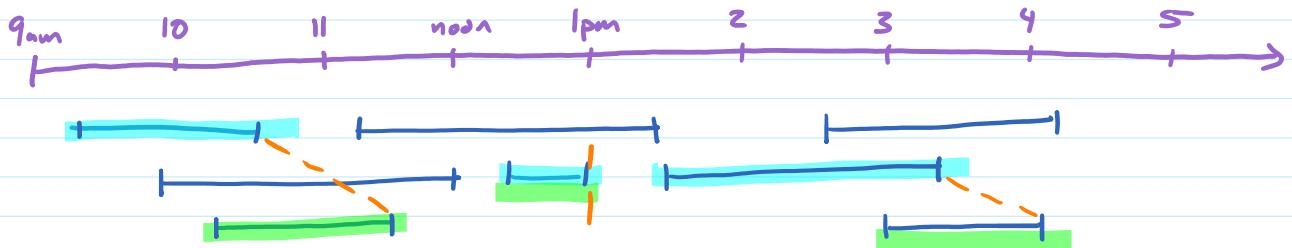
$\Theta$  is sorted

$j_{k+1}$  compatible

INV d

choice of  $i$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE:  $f[1] \leq f[2] \leq \dots \leq f[n]$  and  $s[i] < f[1], \dots, s[n] < f[n]$

POST:  $A$  is a list of distinct indices of non-overlapping intervals that maximizes  $\text{len}(A)$

$A \leftarrow []$

$k \leftarrow 0$

$R \leftarrow \{1, \dots, n\}$

while  $R \neq \emptyset$

choose  $i \in R$  to minimize  $f[i]$

append  $i$  to  $A$

$k \leftarrow k + 1$

remove from  $R$  the intervals  $x$  with  $s[x] < f[i]$

INV: a)  $|A| = k$  and  $A \subseteq R_{in}$

b) intervals in  $A$  are pairwise compatible

c)  $A$  is in order of increasing finish

d)  $R = \{x \in R_{in} \mid s[x] \geq \max_{y \in A} f[y]\}$

e)  $R = \{x \in R_{in} \mid x \text{ is compatible w/ } A\}$

f) for any optimal  $\Theta$  written in order  
of ↑ finish  $j_1, \dots, j_m$

$f[a_k] \leq f[j_i]$  for  $1 \leq i \leq k$

Termination: When  $k=n$ ,  $|A|=R_{in}$  and  $R=\emptyset$

Postcondition: Suppose  $\Theta = j_1, \dots, j_m$  with  $m > k$

Then  $f[a_k] \leq f[j_k]$  INV f

and  $f[j_k] \leq s[j_{k+1}] < f[j_{k+1}]$

so  $s[j_{k+1}] \geq f[a_k]$

and  $j_{k+1} \in R \Rightarrow \Leftarrow$

$\therefore m=k$  and  $A$  is optimal

same size as an optimal solution

