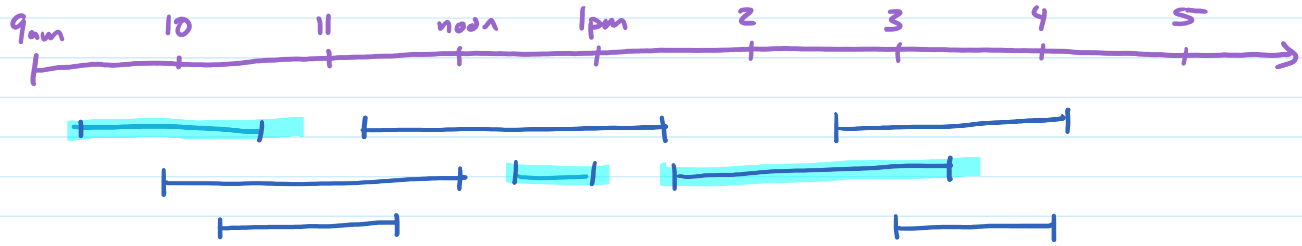
Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



PRE: $f[1] \le f[2] \le \cdots \le f[n]$ and $s[1] < f[1], \ldots, s[n] < f[n]$

POST: A is a list of distinct indices of non-overlapping intervals that maximizes len(A)

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A ← []
k ← 0
R ← {1,...,n}
while R ≠ ∅
    choose i ∈ R to minimize f[i]
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INV:

a) $|A| = k$

b) intervals in A are pairwise compatible

c) A is in order of increasing finish

d) $R = \{ x \in R_{in} \mid s[x] \ge \max_{y \in A} f[y] \}$

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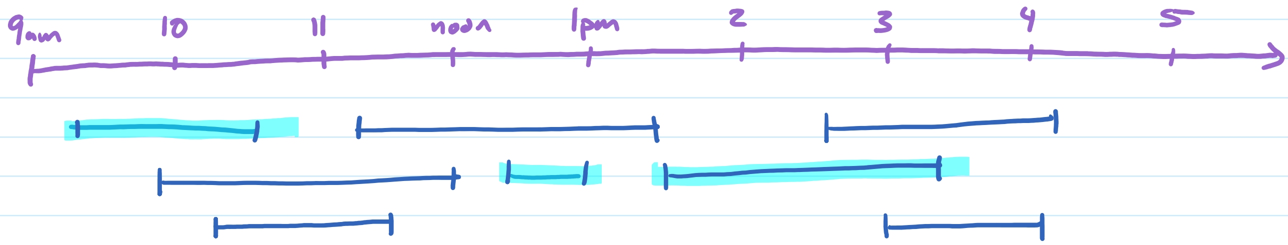

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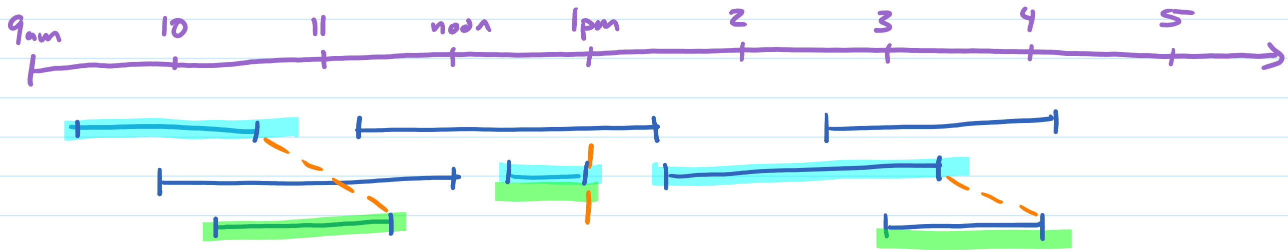

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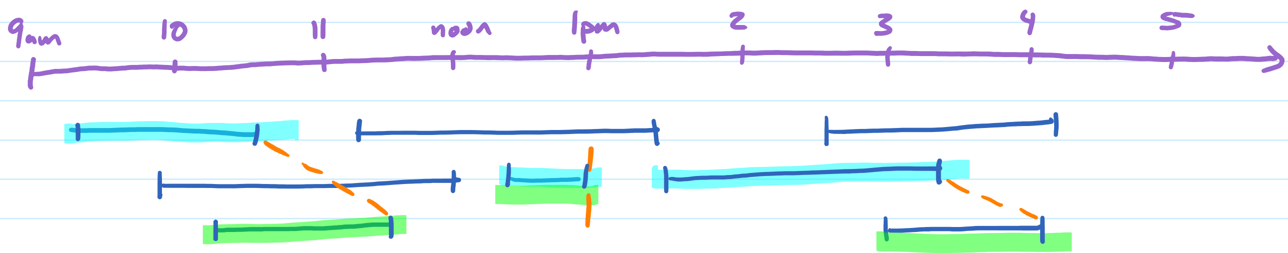

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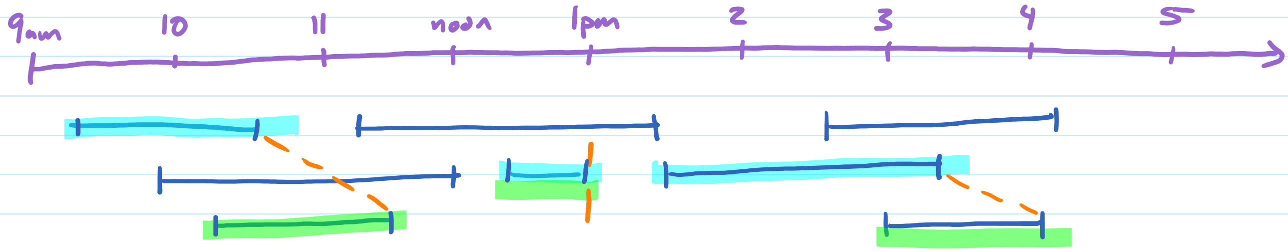

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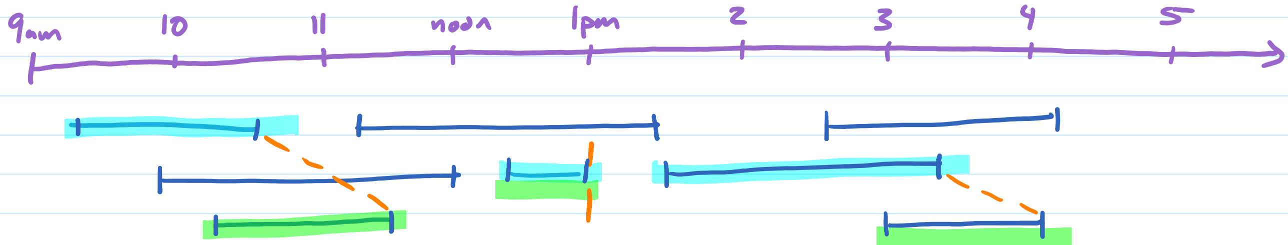
Maintenance:

Interval Scheduing : find largest set of non-overlapping intervals given start/endpoints



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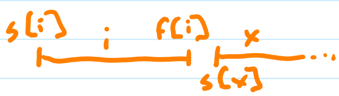
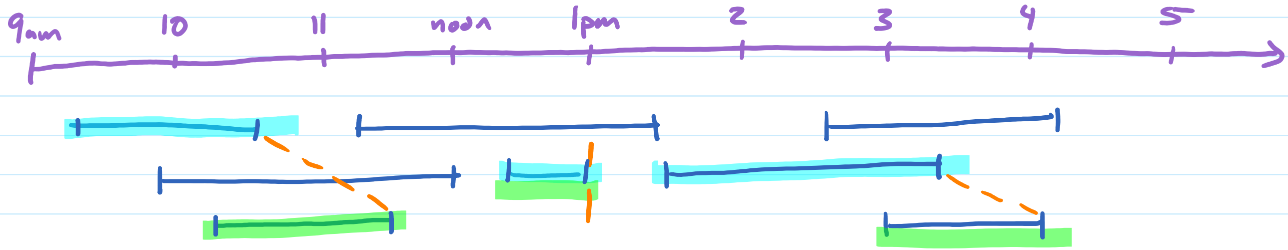
$\subseteq$: Let $x \in R_{new}$.
Then $x \in R_{old}$ — $R_{new} \subseteq R_{old}$
so x is compatible with A_{old} — INV e
Also, $s[x] \ge f[i]$ — choice of R_{new}
So x is compatible with i
$\therefore$ x is compatible with A_{new} — $A_{new} = A_{old} + i$



Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



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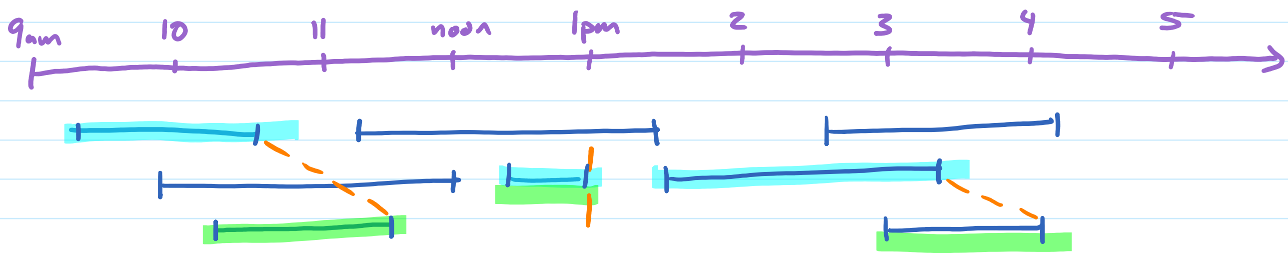

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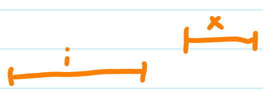

⊇: Suppose x is compatible with A_{new}
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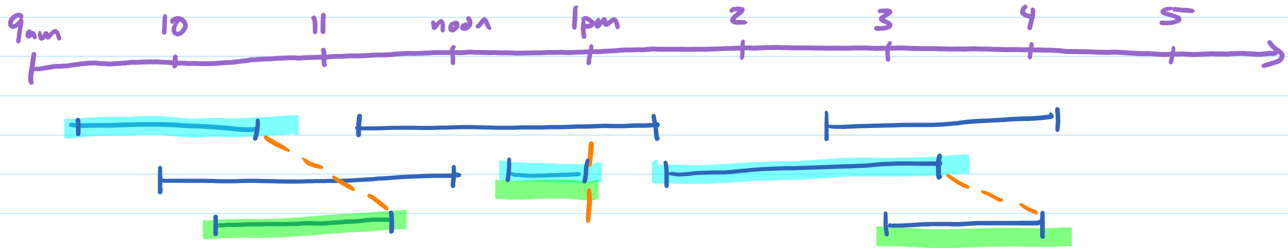

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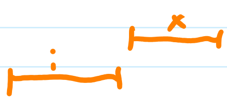

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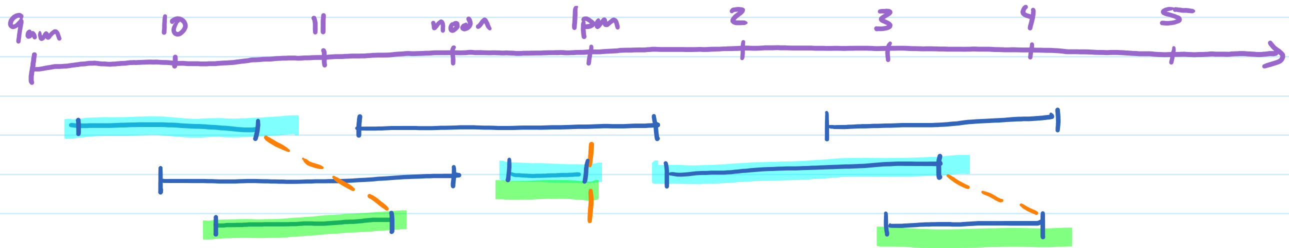

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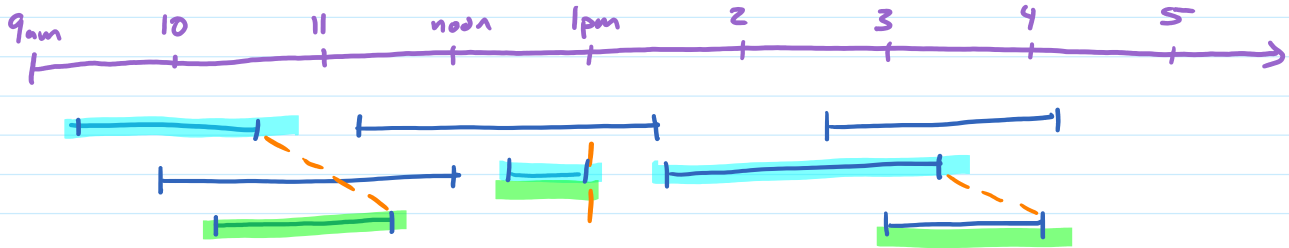
Maintenance: f) 2 cases: i) $k=0$ then $A=[]$ and $R=R_{in}$ and so $f[\underset{i}{\underset{\shortparallel}{a_1}}] \le f[j_1]$ — INV a and e, choice of i

ii) $k>0$

Interval Scheduling : find largest set of non-overlapping intervals given start/endpoints



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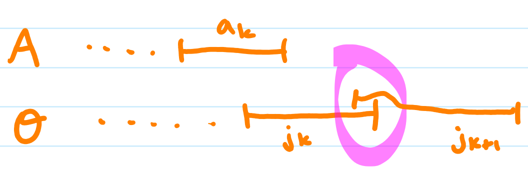
i) $k = 0$ then $A = []$ and $R = R_{in}$ — INV a and e
and so $f[a_1] \le f[j_1]$ — choice of i
$a_1 = i$

ii) $k > 0$ then $f[a_k] \le f[j_k]$ — INV f
Also $f[j_{k+1}] > f[j_k]$ — Θ is sorted



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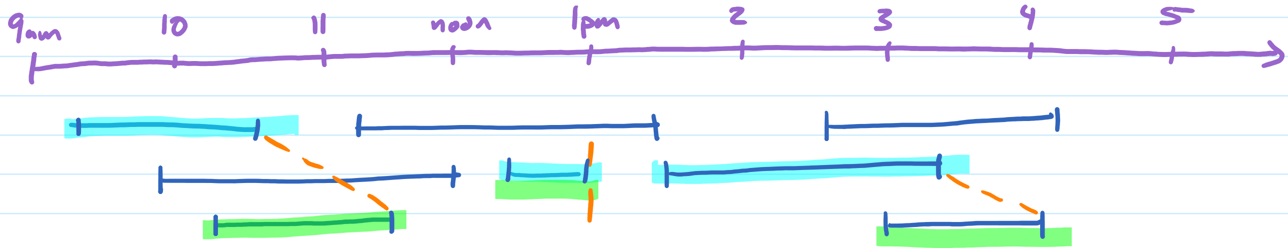

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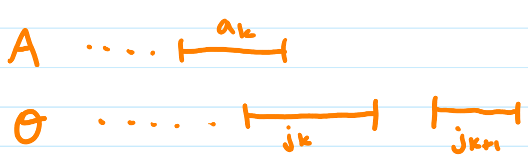

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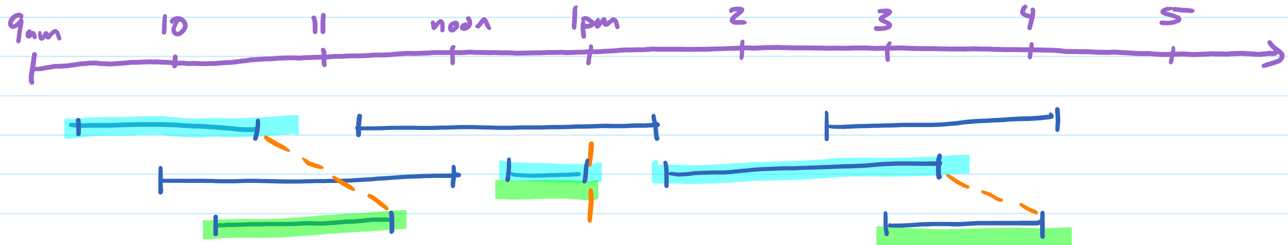

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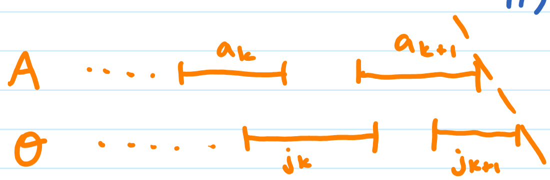
ii) $k>0$ then $f[a_k] \le f[j_k]$ — INV f
Also $f[j_{k+1}] > f[j_k]$ — Θ is sorted
and $s[j_{k+1}] \ge f[j_k]$ — j_k, j_{k+1} compatible
$s[j_{k+1}] \ge f[a_k]$
$j_{k+1} \in R_{old}$ — INV d
$\therefore$ $f[a_{k+1}] \le f[j_{k+1}]$, $a_{k+1} = i$ — choice of i



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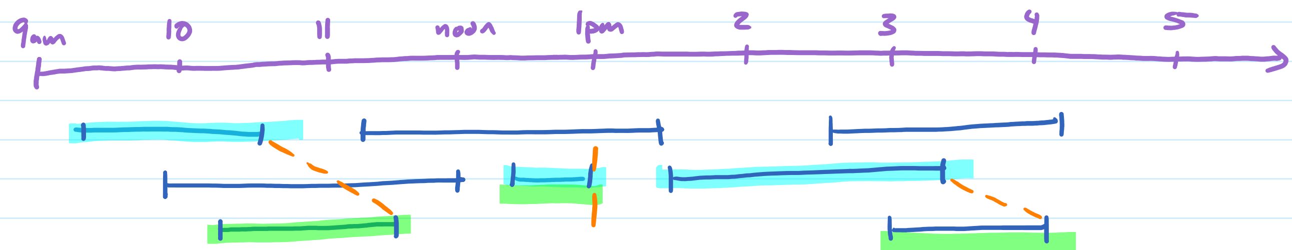

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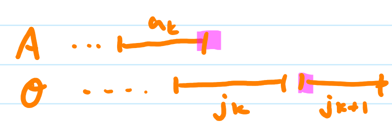
Termination: When $k = n$, $|A| = R_{in}$ and $R = \emptyset$

Postcondition: Suppose $\mathcal{O} = j_1, \ldots, j_m$ with $m > k$

Then $f[a_k] \le f[j_k]$ INV f
and $f[j_k] \le s[j_{k+1}] < f[j_{k+1}]$
so $s[j_{k+1}] \ge f[a_k]$
and $j_{k+1} \in R$ $\Rightarrow\Leftarrow$



$\therefore$ $m = k$ and A is optimal   same size as an optimal solution