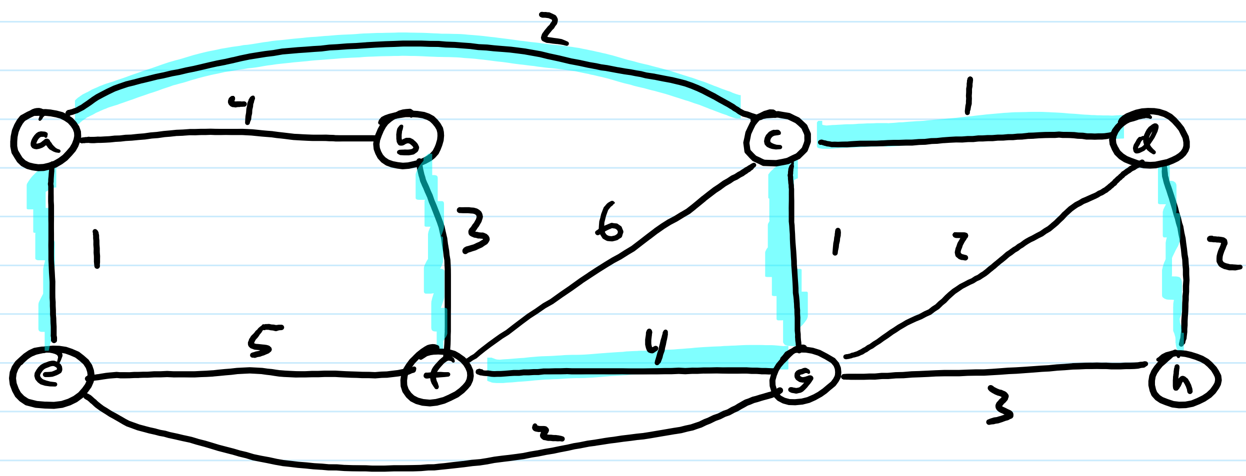
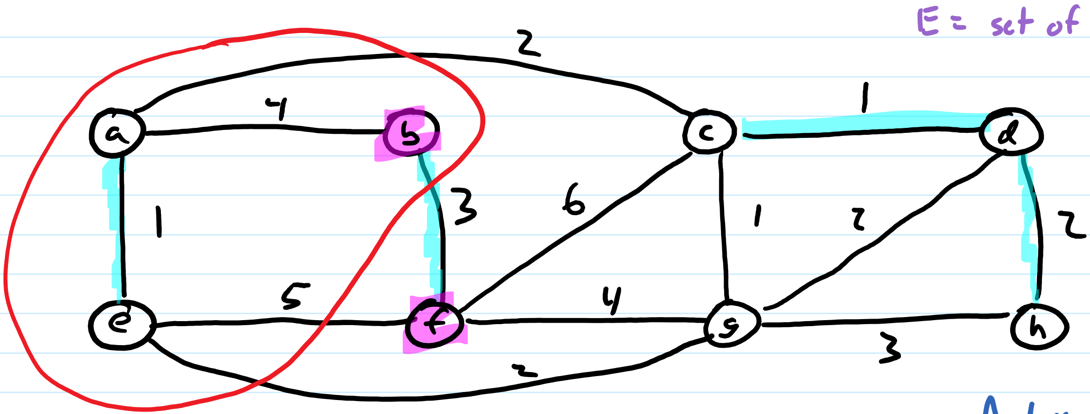
Light Edge Theorem



total weight = 1+2+3+4+1+1+2 = 14

Light Edge Theorem

V = set of all vertices
E = set of all edges



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A does not respect the cut defined by $S = \{a, b, e\}$

cut : a partition of the vertices into S, $V-S$

proto-MST : a subset of E that is also a subset of some MST

a proto-MST A respects a cut S, $V-S$ if every edge in A has both endpoints in S or both endpoints in $V-S$

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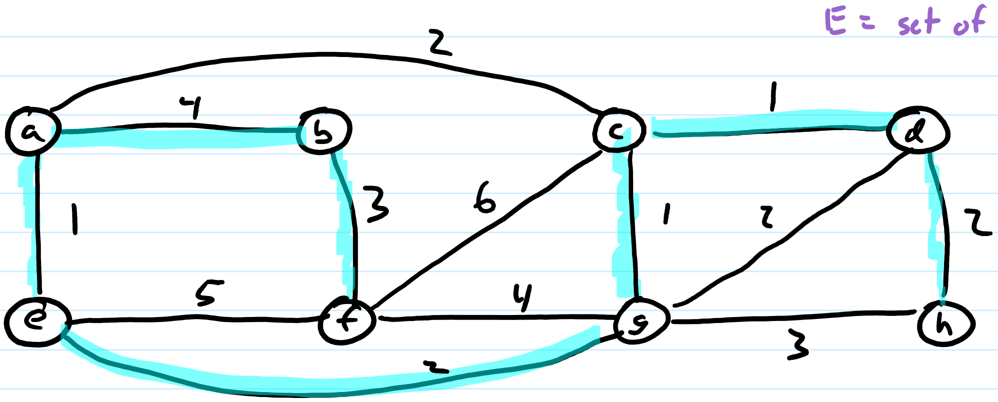

total weight = 1+2+3+4+1+1+2 = 14

THM: If S, V−S is a cut, A is a proto-MST that respects that cut, and (u,v) is an edge of minimum weight across that cut ($u \in S, v \in V-S$ or vice versa), then $A \cup \{(u,v)\}$ is a proto-MST

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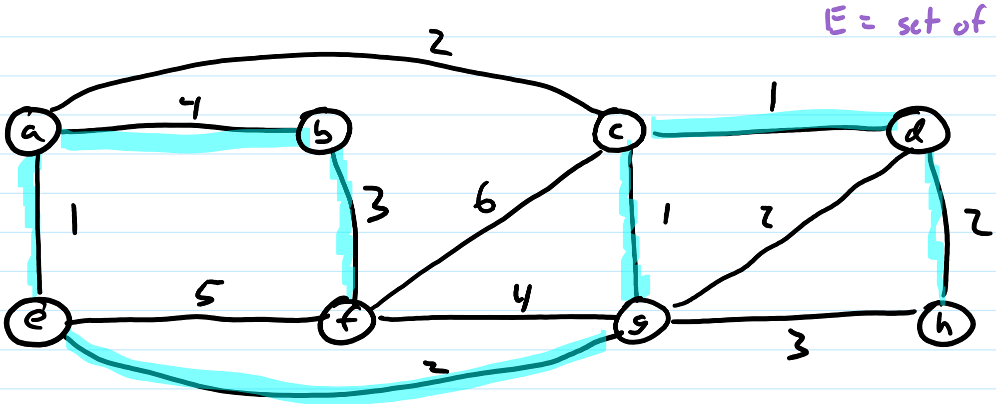

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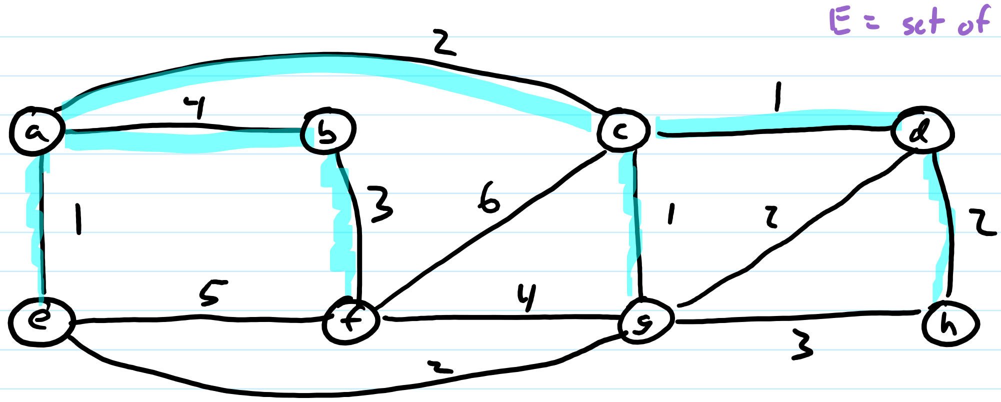

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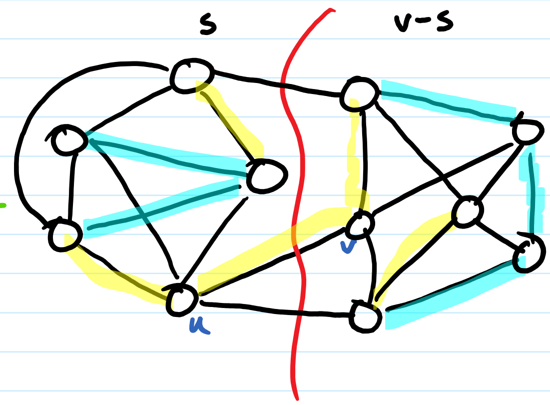
Proof: Let $S, A, (u,v)$ be as given.
Find MST T s.t. $A \subseteq T$ def proto-MST



Two cases: i) $(u,v) \in T$
Then $A \cup \{(u,v)\} \subseteq T$ $A \subseteq T$, $\{(u,v)\} \subseteq T$
so $A \cup \{(u,v)\}$ is a proto-MST

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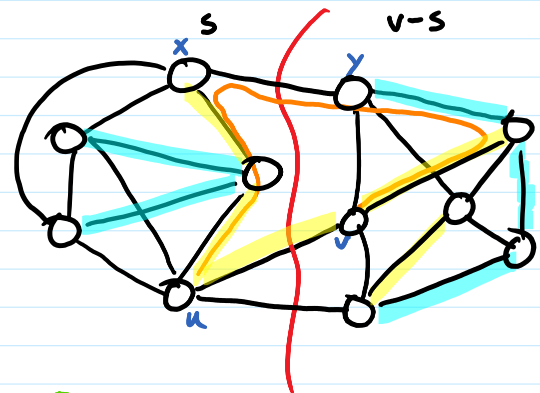
Two cases: ii) $(u,v) \notin T$



$u \rightsquigarrow x \to y \rightsquigarrow v$

Let (x,y) be an edge on path $u \rightsquigarrow v$ in T that crosses cut

Let $T' = T - \{(x,y)\} \cup \{(u,v)\}$

T' connects all vertices (T did; if $v_1 \overset{T}{\rightsquigarrow} v_2$ used (x,y) then $v_1 \rightsquigarrow x \rightsquigarrow u \to v \rightsquigarrow y \rightsquigarrow v_2$ is a path in T')

If $u \to v \rightsquigarrow u$ in T' then $u \rightsquigarrow x \to y \rightsquigarrow v \rightsquigarrow u$ in T

T' is acyclic
T' is a spanning tree

$w(u,v) \le w(x,y)$

$w(T) \le w(T') = w(T) - w(x,y) + w(u,v) \le w(T)$

T is MST

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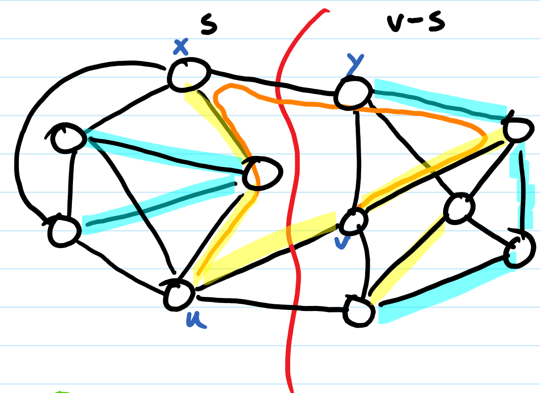
THM: If $S, V-S$ is a cut, A is a proto-MST that respects that cut, and (u,v) is an edge of minimum weight across that cut ($u \in S, v \in V-S$ or vice versa), then $A \cup \{(u,v)\}$ is a proto-MST



Proof: Let $S, A, (u,v)$ be as given.

Find MST T s.t. $A \subseteq T$ def proto-MST

Two cases: ii) $(u,v) \notin T$

$u \rightsquigarrow x \rightarrow y \rightsquigarrow v$

Let (x,y) be an edge on path $u \rightsquigarrow v$ in T that crosses cut

Let $T' = T - \{(x,y)\} \cup \{(u,v)\}$

T' connects all vertices (T did; if $v_1 \overset{T}{\rightsquigarrow} v_2$ used (x,y) then $v_1 \rightsquigarrow x \rightsquigarrow u \rightarrow v \rightsquigarrow y \rightsquigarrow v_2$ is a path in T')

If $u \rightarrow v \rightsquigarrow u$ in T' then $u \rightsquigarrow x \rightarrow y \rightsquigarrow v \rightsquigarrow u$ in T

T' is acyclic

T' is a spanning tree

$w(u,v) \leq w(x,y)$

$w(T) \leq w(T') = w(T) - w(x,y) + w(u,v) \leq w(T)$

T is MST

A respects cut

(x,y) crosses cut

$w(T') = w(T)$

T' is an MST

$A \cup \{(u,v)\} \subseteq T'$

$A \subseteq T - \{(x,y)\} \subseteq T'$ since $(x,y) \notin A$

$(u,v) \in T'$ by choice of T'

$A \cup \{(u,v)\}$ is a proto-MST