Machinist Optimality

m	Pref(m)	w	Pref(w)	Output of Gale-Shapley	Another Stable Matching
A	X Y V W Z	V	A D C E B	(A, Y)	(A, V)
B	V X W Y Z	W	A B D C E	(B, W)	(B, W)
C	V Z W Y X	X	D E C A B	(C, Z)	(C, Y)
D	W V X Z Y	Y	C B A E D	(D, V)	(D, X)
E	X Y V W Z	Z	A B D E C	(E, X)	(E, Z)

DEF: w is a valid partner for m if there is a stable matching S such that $(m,w) \in S$

Y is a valid partner for A V is a valid partner for A

Machinist Optimality

m	Pref m
A	X Y V W Z
B	V X W Y Z
C	V Z W Y X
D	W V X Z Y
E	X Y V W Z

w	Pref w
V	A D C E B
W	A B D C E
X	D E C A B
Y	C B A E D
Z	A B D E C

Output of Gale-Shapley: (A,Y), (B,W), (C,Z), (D,V), (E,X)

Another Stable Matching: (A,V), (B,W), (C,Y), (D,X), (E,Z)

DEF: w is a valid partner for m if there is a stable matching S such that $(m,w) \in S$

Y is a valid partner for A V is a valid partner for A

DEF: best(m) is m's best valid partner - the valid partner of m earliest in m's prefs

m	A	B	C	D	E
best(m)	Y	W	Z	V	X

DEF: stable matching S is machinist optimal if $S = \{ (m, best(M)) \mid m \in M \}$

THM: Gale-Shapley returns the machinist optimal matching

Invariant

a) $\forall m,\ m \notin FreeM \longleftrightarrow \exists w$ s.t. $(m,w) \in Tent$
$\forall w,\ w \notin FreeW \longleftrightarrow \exists m$ s.t. $(m,w) \in Tent$

b) $\forall w,\ w \in FreeW \longleftrightarrow \neg \exists m$ s.t. $(m,w) \in Invites$

FreeM, FreeW keep track of unmatched M, W

free welders are exactly those with no received invitations

c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

d) $|Invites| = k$

e) $\forall w,\ j < k,\ MatchW_j(w) \neq NIL \rightarrow MatchW_{j+1}(w), \ldots, MatchW_k(w) \mathrel{!}= NIL$

The value of MatchW(w) after j iterations

once w receives first invitation, w is never free again

f) $\forall w,\ MatchW(w) = \max_m (m,w) \in Invites$

→ max over w's preference list

the m s.t. $(m,w) \in Tent$ (or NIL if no such m)

(or NIL if no such m) w is matched with most preferred machinist who has sent an invitation to w

g) $\forall m, w, w'$ if $(m,w) \in Invites$ and m prefers w' to w then $(m,w') \in Invites$

machinists send invitations in order of preference

h) $Tent \subseteq Invites$

min over m's preference list

i) $\forall m,\ MatchM(m) \neq NIL \rightarrow MatchM(m) = \min_{(m,w) \in Invites} w$

if a machinist is matched then matched with their last invited w

j) $\forall m,\ (m, best(m)) \in Invites \rightarrow MatchM(m) = best(m)$

m matched with best valid partner if m has sent them an invitation

$\forall m,w\ \ (m,w) \in Invites \rightarrow w = best(m)$ or m prefers w to best(m)

Invariant

a) $\forall m,\ m \notin FreeM \longleftrightarrow \exists w$ s.t. $(m,w) \in Tent$
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$\forall m, w\ \ (m,w) \in Invites \rightarrow w = best(m)$ or m prefers w to best(m)

Suppose $(m,w) \in Invites$ and m prefers best(m) to w
$MatchM(m) = NIL$ or $MatchM(m) = w$ or m prefers w to MatchM(m) — INV i
$(m, best(m)) \in Invites$ — INV g
$MatchM(m) = best(m)$ ⇒⇐ — INV j

Invariant

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$\forall w,\ w \notin FreeW \iff \exists m$ s.t. $(m,w) \in Tent$
FreeM, FreeW keep track of unmatched M, W

b) $\forall w,\ w \in FreeW \iff \neg \exists m$ s.t. $(m,w) \in Invites$
free welders are exactly those with no received invitations

c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

d) $|Invites| = k$

e) $\forall w,\ j < k,\ MatchW_j(w) \neq NIL \rightarrow MatchW_{j+1}(w), \ldots, MatchW_k(w) != NIL$
The value of MatchW(w) after j iterations
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f) $\forall w,\ MatchW(w) = \max_m (m,w) \in Invites$ → max over w's preference list
the m s.t. $(m,w) \in Tent$ (or NIL if no such m)
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i) $\forall m,\ MatchM(m) \neq NIL \rightarrow MatchM(m) = \min_{(m,w) \in Invites} w$
min over m's preference list
if a machinist is matched then matched with their last invited w

j) $\forall m,\ (m, best(m)) \in Invites \rightarrow MatchM(m) = best(m)$
m matched with least valid partner if m has sent them an invitation

Maintenance (j): Suppose the INV is true after k iterations and there is a free m with invitations left

```
while there is an m in FreeM
    choose such an m
    let w be m's next uninvited
    add (m,w) to Invitations

if w in FreeW then
    match m with w
else
    find m' s.t. (m', w) in Tentative
    if w prefers m to m'
        unmatch m' and w
        match m and w
    else
        do nothing
```

Invariant

a) $\forall m,\ m \notin FreeM \iff \exists w$ s.t. $(m,w) \in Tent$
$\forall w,\ w \notin FreeW \iff \exists m$ s.t. $(m,w) \in Tent$
FreeM, FreeW keep track of unmatched M, W

b) $\forall w,\ w \in FreeW \iff \neg\exists m$ s.t. $(m,w) \in Invites$
free welders are exactly those with no received invitations

c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

d) $|Invites| = k$

e) $\forall w,\ j < k,\ MatchW_j(w) \neq NIL \rightarrow MatchW_{j+1}(w), \ldots, MatchW_k(w) != NIL$
The value of MatchW(w) after j iterations
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f) $\forall w,\ MatchW(w) = \max_m (m,w) \in Invites$
→ max over w's preference list
the m s.t. $(m,w) \in Tent$ (or NIL if no such m)
(or NIL if no such m) w is matched with most preferred machinist who has sent an invitation to w

g) $\forall m, w, w'$ if $(m,w) \in Invites$ and m prefers w' to w then $(m,w') \in Invites$
machinists send invitations in order of preference

h) $Tent \subseteq Invites$

i) $\forall m,\ MatchM(m) \neq NIL \rightarrow MatchM(m) = \min_{(m,w) \in Invites} w$
min over m's preference list
if a machinist is matched then matched with their last invited w

j) $\forall m,\ (m, best(m)) \in Invites \rightarrow MatchM(m) = best(m)$
m matched with best valid partner if m has sent them an invitation

Maintenance (j): Suppose the INV is true after k iterations and there is a free m with invitations left

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while there is an m in FreeM
    choose such an m
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else
    find m' s.t. (m', w) in Tentative
    if w prefers m to m'
2a      unmatch m' and w
        match m and w
    else
2b      do nothing
```

For m, in cases 1, 2a

$m \in FreeM_{old}$ choice of m
$MatchM(m) = NIL$
$(m, best(m)) \notin Invites_{old}$ INV j

case i) $w \neq best(m)$
$(m, best(m)) \notin Invites_{new}$

ii) $w = best(m)$
$MatchM(m) = w = best(m)$

Invariant

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$\forall w,\ w \notin FreeW \longleftrightarrow \exists m$ s.t. $(m,w) \in Tent$
FreeM, FreeW keep track of unmatched M, W

b) $\forall w,\ w \in FreeW \longleftrightarrow \neg\exists m$ s.t. $(m,w) \in Invites$
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c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

d) $|Invites| \leq k$

e) $\forall w,\ j < k,\ MatchW_j(w) \neq NIL \rightarrow MatchW_{j+1}(w), \ldots, MatchW_k(w) != NIL$
The value of MatchW(w) after j iterations
once w receives first invitation, w is never free again

f) $\forall w,\ MatchW(w) = \max_m (m,w) \in Invites$
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h) $Tent \subseteq Invites$

i) $\forall m,\ MatchM(m) \neq NIL \rightarrow MatchM(m) = \min_{w,\ (m,w) \in Invites} w$
min over m's preference list
if a machinist is matched then matched with their last invited w

j) $\forall m,\ (m, best(m)) \in Invites \rightarrow MatchM(m) = best(m)$
F T F
m matched with best valid partner if m has sent them an invitation

Maintenance (j): Suppose the INV is true after k iterations and there is a free m with invitations left

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```

For m, in case 2b
Suppose $w = best(m)$
w prefers m' to m
Find stable matching S s.t. $(m,w) \in S$
Find w' s.t. $(m', w') \in S$
w' is a valid partner of m'
$MatchM(m') = w \neq w'$
$w' = best(m')$ or m' prefers $best(m')$ to w'

case 2b
def valid
choice of m'
def best

Invariant

a) $\forall m,\ m \notin FreeM \longleftrightarrow \exists w$ s.t. $(m,w) \in Tent$
$\forall w,\ w \notin FreeW \longleftrightarrow \exists m$ s.t. $(m,w) \in Tent$

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d) $|Invites| = k$

e) $\forall w,\ j < k,\ MatchW_j(w) \neq NIL \rightarrow MatchW_{j+1}(w), \ldots, MatchW_k(w) \mathrel{!}= NIL$

The value of MatchW(w) after j iterations

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h) $Tent \subseteq Invites$

min over m's preference list

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For m, in case 2b

Suppose $w = best(m)$

w prefers m' to m

Find stable matching S s.t. $(m,w) \in S$

Find w' s.t. $(m', w') \in S$

w' is a valid partner of m'

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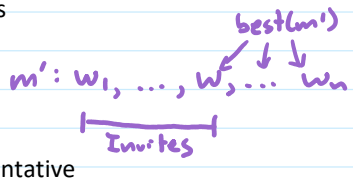

case 2b

def valid choice of m'

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Invariant

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c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

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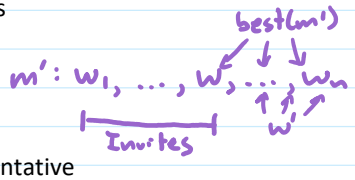
For m, in case 2b

Suppose $w \neq best(m)$



w prefers m' to m

Find stable matching S s.t. $(m,w) \in S$ — case 2b

Find w' s.t. $(m', w') \in S$

w' is a valid partner of m' — def valid

$MatchM(m') = w \neq w'$ — choice of m'

$w' = best(m')$ or m' prefers best(m') to w' — def best

$(m', w') \notin Invites$

$(m', w) \in Invites$ — INV h

m' prefers w to w' — INV g

(m', w) is an instability in S ⇒⇐ — def instability

Invariant

a) $\forall m,\ m \notin FreeM \iff \exists w$ s.t. $(m,w) \in Tent$
$\forall w,\ w \notin FreeW \iff \exists m$ s.t. $(m,w) \in Tent$
} FreeM, FreeW keep track of unmatched M, W

b) $\forall w,\ w \in FreeW \iff \neg\exists m$ s.t. $(m,w) \in Invites$ } free welders are exactly those with no received invitations

c) Tent is a matching and stableish → stable when ignoring unmatched machinists, welders

d) $|Invites| = k$

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The value of MatchW(w) after j iterations

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h) $Tent \subseteq Invites$

i) $\forall m,\ MatchM(m) \neq NIL \rightarrow MatchM(m) = \min_{(m,w) \in Invites} w$ min over m's preference list
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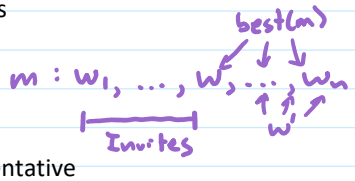

For m', in case 2a
Suppose $w = best(m')$
w prefers m to m'
Find stable matching S s.t. $(m',w) \in S$ case 2b
Find w' s.t. $(m',w') \in S$
w' is a valid partner of m def valid
$MatchM(m) = NIL \neq w'$ choice of m
$w' = best(m)$ or m prefers best(m) to w def best
$(m,w') \notin Invites_{old}$
$(m,w) \in Invites_{new}$ INV h
m prefers w to w' INV g
(m, w) is an instability in S ⇒⇐ def instability

Invariant

a) $\forall m,\ m \notin FreeM \longleftrightarrow \exists w$ s.t. $(m,w) \in Tent$
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   match m and w
    else
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```

Postcondition: Let m be any machinist
find w s.t. $(m,w) \in Tent$
w is a valid partner of m — def valid
suppose $w \neq best(m)$
m prefers best(m) to w — def best
$(m, w) \in Invites$ — INV h
$(m, best(m)) \in Invites$ — INV g
$(m, best(m)) \in Tent$ ⇒⇐ — INV j
$\therefore\ w = best(m)$