Minimizing Maximum Lateness

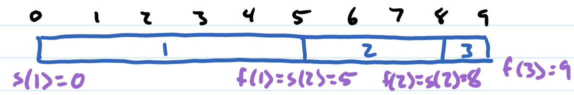
Given tasks with lengths $t_1, \dots, t_n$ and deadlines $d_1, \dots, d_n$, find a schedule that minimizes maximum lateness

once you start a task, you must work on it until complete; can't do 2 tasks at once

start time $s(i)$ for each task so that intervals $(s(i), s(i)+t_i)$ overlap only at endpoints

Example:

task	1	2	3
t_i	5	3	1
d_i	4	8	6



"monus" $\ell_i = f(i) \dot{-} d_i$ $\quad L = \max_i \ell_i$

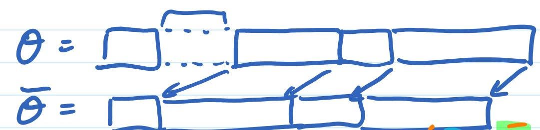
$a \dot{-} b = \begin{cases} a-b & \text{if } a \ge b \\ 0 & \text{otherwise} \end{cases}$

$= \max(0, a-b)$

if $a > b$ then $x-a < x-b$

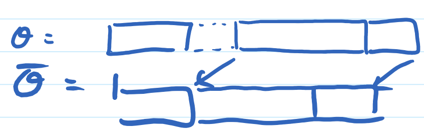
if $a > b$ then $x \dot{-} a \le x \dot{-} b$

1) There is an optimal solution w/ no idle time



$\max(\bar{\ell}_1, \bar{\ell}_2, \dots, \bar{\ell}_n) \le \max(\ell_1, \dots, \ell_n)$

1) There is an optimal schedule with no idle time



$\bar{f}(i) = f(i) -$ idle time before i in schedule for all i

so $\bar{f}(i) \le f(i)$

so $\bar{\ell}_i = \bar{f}(i) - d_i \le f(i) - d_i = \ell_i$

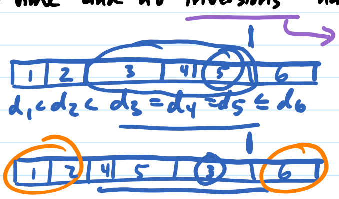
and $\bar{L} = \max_i(\bar{\ell}_i) \le \max_i(\ell_i) = L$

so $\bar{\theta}$ is still optimal

2) All schedules with no idle time and no inversions have same max lateness

inversions → task i before task j but $d_i > d_j$

$\max(\ell_3, \ell_4, \ell_5) = \ell_5 = f(5) - d_5$



$\max(\bar{\ell}_3, \bar{\ell}_4, \bar{\ell}_5) = \bar{\ell}_3 = \bar{f}(3) - d_3$

$= f(5) - d_5$

$L = \max(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6)$

$= \max(\ell_1, \ell_2, \ell_6, \max(\ell_3, \ell_4, \ell_5))$

$= \max(\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_6, \max(\bar{\ell}_3, \bar{\ell}_4, \bar{\ell}_5))$

$= \max(\bar{\ell}_1, \dots, \bar{\ell}_6) = \bar{L}$

$\max(\bar{l}_3, \bar{l}_4, \bar{l}_5) = \bar{l}_3 = \bar{f}(3) - d_3$ | 1 | 2 | 4 | 5 | 3 | 6 |
$= f(5) - d_5$
$= l_5 = \max(l_3, l_4, l_5)$

$= \max(\bar{l}_1, \bar{l}_2, \bar{l}_6, \max(\bar{l}_3, \bar{l}_4, \bar{l}_5))$
$= \max(\bar{l}_1, \ldots, \bar{l}_6) = \bar{L}$

3) There is a optimal schedule with no inversions, no idle time.

So greedy has no inversions, no idle time:
There is optimal Θ w/ no inversions, no idle (3)
greedy has same max lateness as Θ (2)
$\therefore$ greedy is optimal — same max lateness

Suppose not - that is, all optimal Θ with no idle time have inversions
Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in Θ so i before j but $d_i > d_j$
Assume WLOG that j follows i immediately in Θ

(let x be index of i in Θ so $\Theta_x = i$
let y be index of j in Θ so $\Theta_y = j$
can't have $d_{\Theta_x} \le d_{\Theta_{x+1}} \le \ldots \le d_{\Theta_y}$
since then $d_i = d_{\Theta_x} \le d_{\Theta_y} = d_j$ and
i, j is not an inversion)

Construct $\bar{\Theta}$ by swapping (exchanging) i and j:

Θ: | X | i | j | |

$\Downarrow$

$\bar{\Theta}$: | X | j | i | |

(still no idle time)

"monus"
$$x \dot{-} y = \begin{cases} 0 & \text{if } y > x \\ x - y & \text{otherwise} \end{cases}$$

if $a > b$ then $x - a < x - b$ and $x \dot{-} a \le x \dot{-} b$

$l_j = f(j) \dot{-} d_j$

$$l_i = f(i) \dot{-} d_i \le f(j) \dot{-} d_i \le \underline{f(j) \dot{-} d_j} = l_j$$

def lateness; $f(j) > f(i)$ since j after i; $d_i > d_j$ since inverted; def lateness

so $\max(l_i, l_j) = l_j$

lateness of j went down since it was moved up

$$\bar{l}_j = \bar{f}(j) \dot{-} d_j \le f(j) \dot{-} d_j = l_j \quad \text{so } \bar{l}_j \le l_j$$

def lateness; $\bar{f}(j) \le f(j)$ since j moved up; def lateness

new lateness of i not worse than lateness of j

$$\bar{l}_i = \bar{f}(i) \dot{-} d_i = f(j) \dot{-} d_i \le f(j) \dot{-} d_j = l_j$$

def lateness; $\bar{f}(i) = f(j)$ b/c; $d_i > d_j$; def lateness

$f(j) = \sum_{x \in X} t_x + t_i + t_j$
$\bar{f}(i) = \sum_{x \in X} t_x + t_j + t_i$

finish = sum of times of tasks before + own time

$$\max(\overline{l}_i, \overline{l}_j) \le \max(l_j, l_j) = l_j \le \max(l_i, l_j)$$

$$\max(\overline{l}_1, \ldots, \overline{l}_n) = \max\left(\max_{k \ne i,j} \overline{l}_k, \max(\overline{l}_i, \overline{l}_j)\right)$$

$$= \max\left(\max_{k \ne i,j} l_k, \max(\overline{l}_i, \overline{l}_j)\right)$$ lateness of tasks other than i,j didn't change

$$\le \max\left(\max_{k \ne i,j} l_k, \max(l_i, l_j)\right)$$ max of larger things is larger

$$= \max(l_1, \ldots, l_n)$$

$$\max(l_1, \ldots, l_n) \le \max(\overline{l}_1, \ldots, \overline{l}_n)$$ Θ is optimal

$$\max(\overline{l}_1, \ldots, \overline{l}_n) = \max(l_1, \ldots, l_n)$$ $\ge$ and $\le$

$\therefore \overline{\Theta}$ is optimal   same max lateness as opt

$\overline{\Theta}$ has fewer inversions than Θ fixing i,j makes count ↓ 1

$\not\Rightarrow$

so there is an optimal Θ with no inversions and no idle time

Minimizing Maximum Lateness

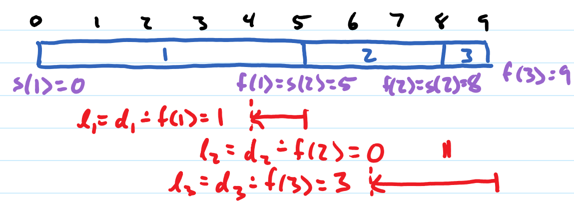
Given tasks with lengths $t_1, \dots, t_n$ and deadlines $d_1, \dots, d_n$, find a schedule that minimizes maximum lateness

once you start a task, you must work on it until complete; can't do 2 tasks at once

{start time $s(i)$ for each task so that intervals $(s(i), s(i)+t_i)$ overlap only at endpoints

Example:

task	1	2	3
t_i	5	3	1
d_i	4	8	6



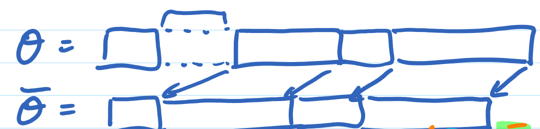
"monus" $\ell_i = f(i) \dot{-} d_i$ $\quad L = \max_i \ell_i$

$$a \dot{-} b = \begin{cases} a-b & \text{if } a \ge b \\ 0 & \text{otherwise} \end{cases} = \max(0, a-b)$$

if $a > b$ then $x - a < x - b$

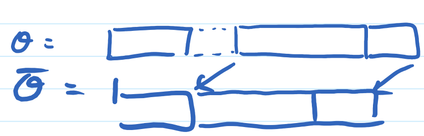
if $a > b$ then $x \dot{-} a \le x \dot{-} b$

1) There is an optimal solution w/ no idle time



$\max(\bar{\ell}_1, \bar{\ell}_2, \dots, \bar{\ell}_n) \le \max(\ell_1, \dots, \ell_n)$

1) There is an optimal schedule with no idle time



$\bar{f}(i) = f(i) -$ idle time before i in schedule for all i

so $\bar{f}(i) \le f(i)$

so $\bar{\ell}_i = \bar{f}(i) - d_i \le f(i) - d_i = \ell_i$

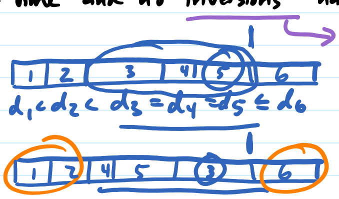
and $\bar{L} = \max_i(\bar{\ell}_i) \le \max_i(\ell_i) = L$

so $\bar{\theta}$ is still optimal

2) All schedules with no idle time and no inversions have same max lateness

task i before task j but $d_i > d_j$

$\max(\ell_3, \ell_4, \ell_5) = \ell_5 = f(5) - d_5$



$\max(\bar{\ell}_3, \bar{\ell}_4, \bar{\ell}_5) = \bar{\ell}_3 = \bar{f}(3) - d_3$

$= f(5) - d_5$

$L = \max(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6)$

$= \max(\ell_1, \ell_2, \ell_6, \max(\ell_3, \ell_4, \ell_5))$

$= \max(\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_6, \max(\bar{\ell}_3, \bar{\ell}_4, \bar{\ell}_5))$

$= \max(\bar{\ell}_1, \dots, \bar{\ell}_6) = \bar{L}$

$\max(\bar{l}_3, \bar{l}_4, \bar{l}_5) = \bar{l}_3 = \bar{f}(3) - d_3$
$= f(5) - d_5$
$= l_5 = \max(l_3, l_4, l_5)$

$= \max(\bar{l}_1, \bar{l}_2, \bar{l}_6, \max(\bar{l}_3, \bar{l}_4, \bar{l}_5))$
$= \max(\bar{l}_1, \ldots, \bar{l}_6) = \bar{L}$

3) There is a optimal schedule with no inversions, no idle time.

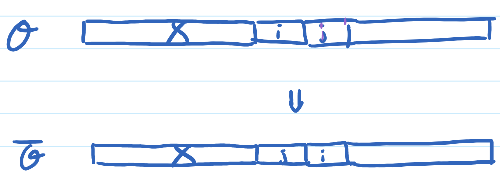
So greedy has no inversions, no idle time:
There is optimal Θ w/ no inversions, no idle (3)
greedy has same max lateness as Θ (2)
$\therefore$ greedy is optimal
same max lateness

Suppose not - that is, all optimal Θ with no idle time have inversions
Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in Θ so i before j but $d_i > d_j$
Assume WLOG that j follows i immediately in Θ

(let x be index of i in Θ so $\Theta_x = i$
let y be index of j in Θ so $\Theta_y = j$
can't have $d_{\Theta_x} \le d_{\Theta_{x+1}} \le \ldots \le d_{\Theta_y}$
since then $d_i = d_{\Theta_x} \le d_{\Theta_y} = d_j$ and
i, j is not an inversion)

Construct $\bar{\Theta}$ by swapping (exchanging) i and j:



(still no idle time)

"monus"
$$x \dot{-} y = \begin{cases} 0 & \text{if } y > x \\ x - y & \text{otherwise} \end{cases}$$

if $a > b$ then $x - a < x - b$ and $x \dot{-} a \le x \dot{-} b$

$l_j = f(j) \dot{-} d_j$

$$l_i = f(i) \dot{-} d_i \le f(j) \dot{-} d_i \le \underline{f(j) \dot{-} d_j} = l_j$$

def lateness; $f(j) > f(i)$ since j after i; $d_i > d_j$ since inverted; def lateness

so $\max(l_i, l_j) = l_j$

lateness of j went down since it was moved up

$$\bar{l}_j = \bar{f}(j) \dot{-} d_j \le f(j) \dot{-} d_j = l_j \quad \text{so } \bar{l}_j \le l_j$$

def lateness; $\bar{f}(j) \le f(j)$ since j moved up; def lateness

new lateness of i not worse than lateness of j

$$\bar{l}_i = \bar{f}(i) \dot{-} d_i = f(j) \dot{-} d_i \le f(j) \dot{-} d_j = l_j$$

def lateness; $\bar{f}(i) = f(j)$ b/c; $d_i > d_j$; def lateness

$f(j) = \sum_{x \in X} t_x + t_i + t_j$
$\quad\quad \shortparallel$
$\bar{f}(i) = \sum_{x \in X} t_x + t_j + t_i$

finish = sum of times of tasks before + own time

$\max(\bar{\ell}_i, \bar{\ell}_j) \leq \max(\ell_j, \ell_j) = \ell_j \leq \max(\ell_i, \ell_j)$

$\max(\bar{\ell}_1, \dots, \bar{\ell}_n) = \max\left(\max_{k \neq i,j} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right)$

$= \max\left(\max_{k \neq i,j} \ell_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right)$ lateness of tasks other than i,j didn't change

$\leq \max\left(\max_{k \neq i,j} \ell_k, \max(\ell_i, \ell_j)\right)$ max of larger things is larger

$= \max(\ell_1, \dots, \ell_n)$

$\max(\ell_1, \dots, \ell_n) \leq \max(\bar{\ell}_1, \dots, \bar{\ell}_n)$ $\quad \theta$ is optimal

$\max(\bar{\ell}_1, \dots, \bar{\ell}_n) = \max(\ell_1, \dots, \ell_n)$ $\quad \geq$ and $\leq$

$\therefore$ $\bar{\theta}$ is optimal $\quad$ same max lateness as opt

$\bar{\theta}$ has fewer inversions than θ $\quad$ fixing i,j makes count $\downarrow 1$

$\Longrightarrow$

so there is an optimal θ with no inversions and no idle time

Minimizing Maximum Lateness

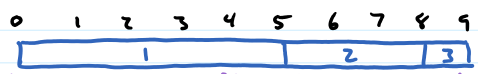
Given tasks with lengths $t_1, \ldots, t_n$ and deadlines $d_1, \ldots, d_n$, find a schedule that minimizes maximum lateness

once you start a task, you must work on it until complete; can't do 2 tasks at once

{ start time $s(i)$ for each task so that intervals $(s(i), s(i)+t_i)$ overlap only at endpoints

Example:

task	1	2	3
t_i	5	3	1
d_i	4	8	6



$s(1)=0$ $\quad f(1)=s(2)=5$ $\quad f(2)=s(3)=8$ $\quad f(3)=9$

$\ell_1 = f(1) - d_1 = 1$

$\ell_2 = f(2) \dot{-} d_2 = 0$

$\ell_3 = f(3) \dot{-} d_3 = 3$

$L = \max(1, 0, 3) = 3$

"monus" $\quad \ell_i = f(i) \dot{-} d_i \qquad L = \max_i \ell_i$

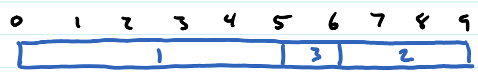
$a \dot{-} b = \begin{cases} a-b & \text{if } a \geq b \\ 0 & \text{otherwise} \end{cases}$

$= \max(0, a-b)$

if $a > b$ then $x - a < x - b$

if $a > b$ then $x \dot{-} a \leq x \dot{-} b$

earliest deadline first



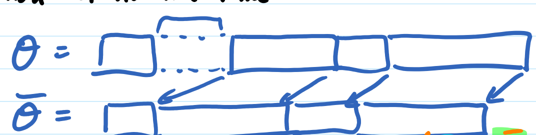
$\ell_1 = 5 \dot{-} 4 = 1$

$\ell_2 = 6 \dot{-} 6 = 0$

$\ell_3 = 9 \dot{-} 8 = 1$

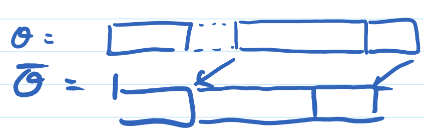
$L = 1$

1) There is an optimal solution w/ no idle time



$\max(\bar{\ell}_1, \bar{\ell}_2, \ldots, \bar{\ell}_n) \leq \max(\ell_1, \ell_2, \ldots, \ell_n)$

1) There is an optimal schedule with no idle time



$\bar{f}(i) = f(i) -$ idle time before i in schedule for all i

so $\bar{f}(i) \leq f(i)$

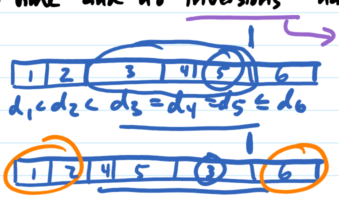
so $\bar{\ell}_i = \bar{f}(i) \dot{-} d_i \leq f(i) \dot{-} d_i = \ell_i$

and $\bar{L} = \max_i(\bar{\ell}_i) \leq \max_i(\ell_i) = L$

so $\bar{\mathcal{O}}$ is still optimal

2) All schedules with no idle time and no inversions have same max lateness

→ task i before task j but $d_i > d_j$



$\max(\ell_3, \ell_4, \ell_5) = \ell_5 = f(5) - d_5$

$d_1 < d_2 < d_3 = d_4 = d_5 \leq d_6$

$\max(\bar{\ell}_3, \bar{\ell}_4, \bar{\ell}_5) = \bar{\ell}_3 = \bar{f}(3) - d_3$

$= f(5) - d_5$

$L = \max(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6)$

$= \max(\ell_1, \ell_2, \ell_6, \max(\ell_3, \ell_4, \ell_5))$

$= \max(\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_6, \max(\bar{\ell}_3, \bar{\ell}_4, \bar{\ell}_5))$

$= \max(\bar{\ell}_1, \ldots, \bar{\ell}_6) = \bar{L}$

$\max(\bar{l}_3, \bar{l}_4, \bar{l}_5) = l_3 = f(3) - d_3$
$= f(5) - d_5$
$= l_5 = \max(l_3, l_4, l_5)$

$= \max(\bar{l}_1, \bar{l}_2, \bar{l}_6, \max(\bar{l}_3, \bar{l}_4, \bar{l}_5))$
$= \max(\bar{l}_1, \ldots, \bar{l}_6) = \bar{L}$

3) There is a optimal schedule with no inversions, no idle time.

So greedy has no inversions, no idle time:
There is optimal Θ w/ no inversions, no idle (3)
greedy has same max lateness as Θ (2)
$\therefore$ greedy is optimal

same max lateness

Suppose not - that is, all optimal Θ with no idle time have inversions
Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

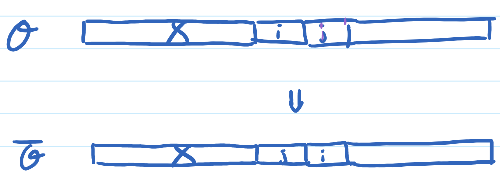
Let i, j be an inversion in Θ so i before j but $d_i > d_j$
Assume WLOG that j follows i immediately in Θ

(let x be index of i in Θ so $\Theta_x = i$
let y be index of j in Θ so $\Theta_y = j$
can't have $d_{\Theta_x} \le d_{\Theta_{x+1}} \le \ldots \le d_{\Theta_y}$
since then $d_i = d_{\Theta_x} \le d_{\Theta_y} = d_j$ and
i, j is not an inversion)

Construct $\bar{\Theta}$ by swapping (exchanging) i and j:

"monus"
$$x \dot{-} y = \begin{cases} 0 & \text{if } y > x \\ x - y & \text{otherwise} \end{cases}$$

if $a > b$ then $x - a < x - b$
and $x \dot{-} a \le x \dot{-} b$



(still no idle time)

$l_j = f(j) \dot{-} d_j$

$$l_i = f(i) \dot{-} d_i \le f(j) \dot{-} d_i \le f(j) \dot{-} d_j = l_j$$

def lateness — $f(j) > f(i)$ since j after i — $d_i > d_j$ since inverted — def lateness

so $\max(l_i, l_j) = l_j$

lateness of j went down since it was moved up

$$\bar{l}_j = \bar{f}(j) \dot{-} d_j \le f(j) \dot{-} d_j = l_j \quad \text{so } \bar{l}_j \le l_j$$

def lateness — $\bar{f}(j) \le f(j)$ since j moved up — def lateness

new lateness of i not worse than lateness of j

$$\bar{l}_i = \bar{f}(i) \dot{-} d_i = f(j) \dot{-} d_i \le f(j) \dot{-} d_j = l_j$$

def lateness — $\bar{f}(i) = f(j)$ b/c — $d_i > d_j$ — def lateness

$f(j) = \sum_{x \in X} t_x + t_i + t_j$
$\bar{f}(i) = \sum_{x \in X} t_x + t_j + t_i$

finish = sum of times of tasks before + own time

$\max(\bar{\ell}_i, \bar{\ell}_j) \le \max(\ell_j, \ell_j) = \ell_j \le \max(\ell_i, \ell_j)$

$$\max(\bar{\ell}_1, \dots, \bar{\ell}_n) = \max\left(\max_{k \ne i,j} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right)$$

$$= \max\left(\max_{k \ne i,j} \ell_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right)$$ lateness of tasks other than i,j didn't change

$$\le \max\left(\max_{k \ne i,j} \ell_k, \max(\ell_i, \ell_j)\right)$$ max of larger things is larger

$$= \max(\ell_1, \dots, \ell_n)$$

$\max(\ell_1, \dots, \ell_n) \le \max(\bar{\ell}_1, \dots, \bar{\ell}_n)$ $\quad$ Θ is optimal

$\max(\bar{\ell}_1, \dots, \bar{\ell}_n) = \max(\ell_1, \dots, \ell_n)$ $\quad$ $\ge$ and $\le$

$\therefore$ $\bar{\Theta}$ is optimal $\quad$ same max lateness as opt

$\bar{\Theta}$ has fewer inversions than Θ $\quad$ fixing i,j makes count $\downarrow$ 1

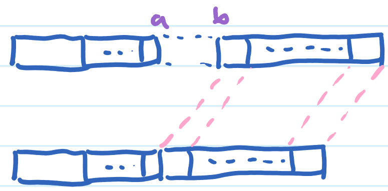
$\Rightarrow\Leftarrow$

so there is an optimal Θ with no inversions and no idle time

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

If optimal Θ has idle time



Construct new $\bar{\Theta}$ with none

$\bar{L} \leq L$, so $\bar{\Theta}$ is also optimal

Suppose all optimal solutions have idle time.

Find optimal Θ with fewest idle periods, let k be the number of idle periods, and let (a, b) be an idle period in Θ.

Construct $\bar{\Theta}$ by setting $\bar{s}(i) = \begin{cases} s(i) & \text{if } s(i) < a \\ s(i) - (b-a) & \text{otherwise} \end{cases}$

move the tasks after idle up to eliminate the idle time

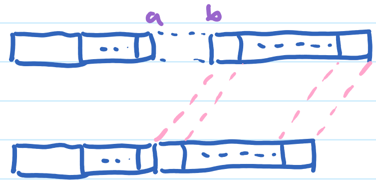
Now $\bar{f}(i) = \bar{s}(i) + t_i \leq s(i) + t_i = f(i)$ for all i — choice of $\bar{s}$

and $\bar{\ell}_i = \bar{f}(i) \dot{-} d_i \leq f(i) \dot{-} d_i = \ell_i$ for all i — def lateness

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

If optimal θ has idle time



Construct new $\bar{\theta}$ with none

$\bar{L} \le L$, so $\bar{\theta}$ is also optimal

~~Suppose all optimal solutions have idle time.~~

Find optimal θ with fewest idle periods, let k be the number of idle periods, and let (a,b) be an idle period in θ.

Construct $\bar{\theta}$ by setting $\bar{s}(i) = \begin{cases} s(i) & \text{if } s(i) < a \\ s(i) - (b-a) & \text{otherwise} \end{cases}$ — move the tasks after idle up to eliminate the idle time

Now $\bar{f}(i) = \bar{s}(i) + t_i \le s(i) + t_i = f(i)$ for all i — choice of $\bar{s}$

and $\bar{\ell}_i = \bar{f}(i) \dot{-} d_i \le f(i) \dot{-} d_i = \ell_i$ for all i — def lateness

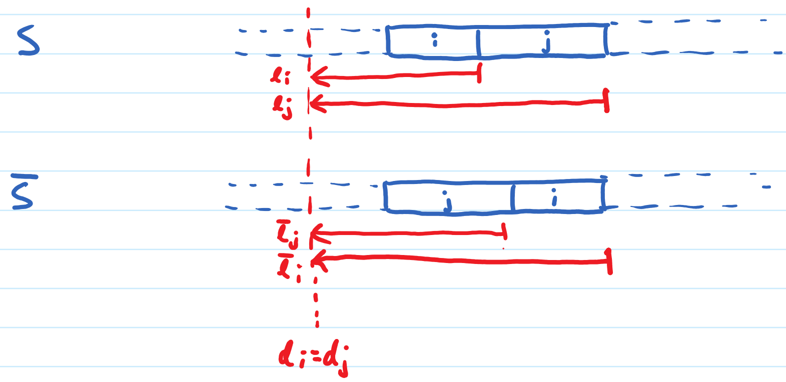
so $\bar{L} = \max_i \bar{\ell}_i \le \max_i \ell_i = L$ so $\bar{\theta}$ is also optimal and has fewer idle periods than θ $\Rightarrow\Leftarrow$

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

two tasks a, b not in order of deadline $\left(\begin{matrix} s(a) < s(b) \\ \text{but} \\ d_a > d_b \end{matrix} \right)$

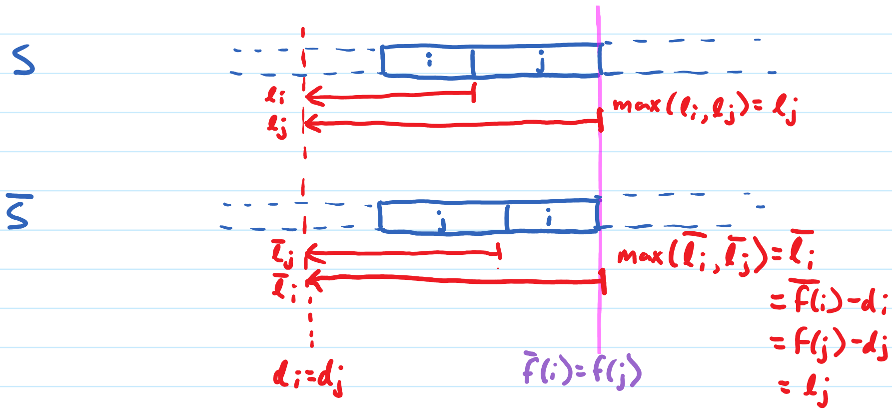


Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

two tasks a, b not in order of deadline $\left(\begin{array}{c} s(a) < s(b) \\ \text{but} \\ d_a > d_b \end{array}\right)$

2) All schedules with no idle time and no inversions have the same maximum lateness



$$L = \max_k \ell_k$$

$$= \max\left(\max_{\substack{k \neq i \\ k \neq j}} \ell_k, \max(\ell_i, \ell_j)\right)$$

$$= \max\left(\max_{\substack{k \neq i \\ k \neq j}} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right)$$

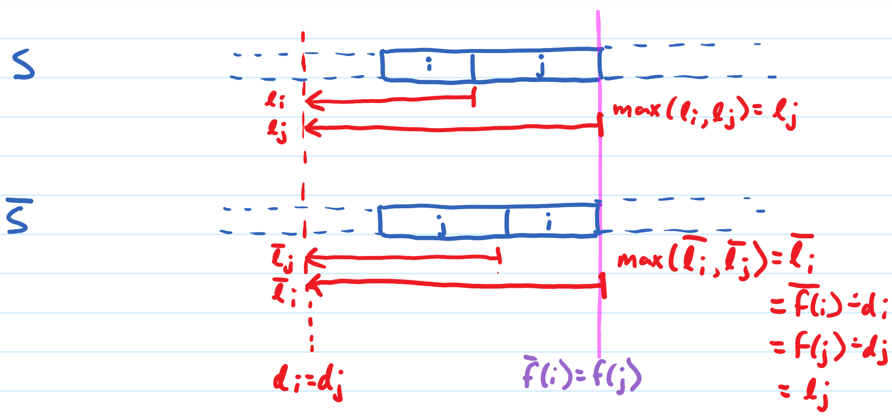
$$= \bar{L}$$

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

two tasks a, b not in order of deadline $\left(s(a) < s(b) \text{ but } d_a > d_b \right)$

2) All schedules with no idle time and no inversions have the same maximum lateness



$$L = \max_k l_k$$
$$= \max\left(\max_{\substack{k \neq i \\ k \neq j}} l_k,\ \max(l_i, l_j)\right)$$
$$= \max\left(\max_{\substack{k \neq i \\ k \neq j}} \bar{l}_k,\ \max(\bar{l}_i, \bar{l}_j)\right)$$
$$= \bar{L}$$

If $S, \bar{S}$ both have no idle time and no inversions then transform S into $\bar{S}$ by swapping adjacent tasks with same deadline one at a time

One such swap doesn't change maximum lateness; prove by induction that n such swaps don't, so $S, \bar{S}$ have same maximum lateness.

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

THM: The greedy (earliest deadline first) schedule G is optimal

Proof: G has no inversions and no idle time — construction of G

There is an optimal O with no inversions or idle time — 3

G and O have same maximum lateness — 2

G is also optimal — same max lateness as optimal

Minimizing Maximum Lateness

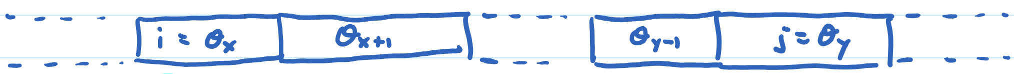
1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

Suppose not - that is, all optimal $\mathcal{O}$ with no idle time have inversions
Pick optimal $\mathcal{O}$ with fewest inversions (well-ordering)
and no idle time (1)

Let i, j be an inversion in $\mathcal{O}$ so i before j but $d_i > d_j$
Assume WLOG that j follows i immediately in $\mathcal{O}$



$$d_i = d_{\mathcal{O}_x} \le d_{\mathcal{O}_{x+1}} \le d_{\mathcal{O}_{x+2}} \le \dots \le d_{\mathcal{O}_{y-1}} \le d_{\mathcal{O}_y} = d_j$$

$d_i \le d_j$ so i, j isn't an inversion

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

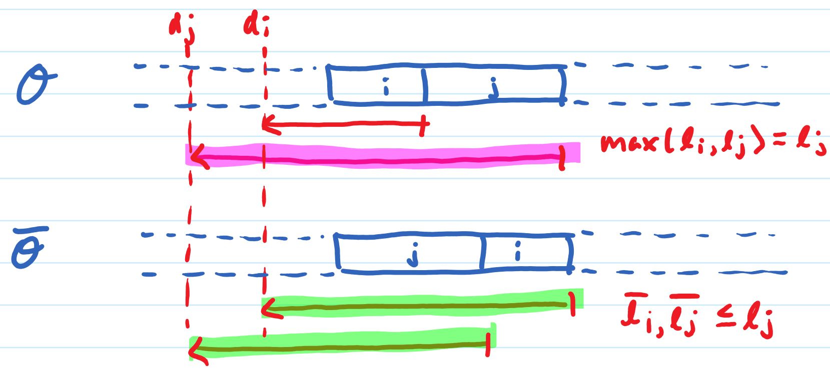
Suppose not - that is, all optimal Θ with no idle time have inversions

Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in Θ so i before j but $d_i > d_j$

Assume WLOG that j follows i immediately in Θ

Construct $\overline{\Theta}$ by swapping (exchanging) i and j:



Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

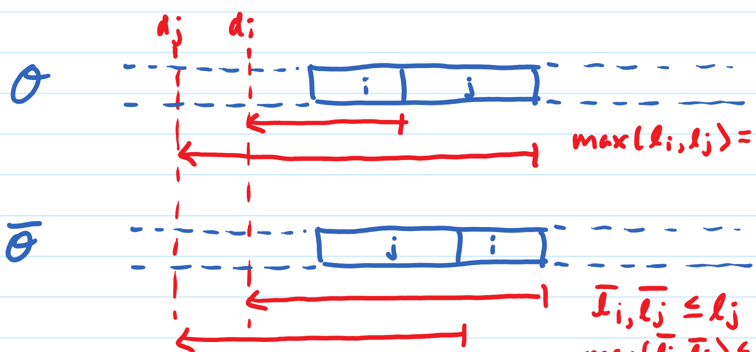
Suppose not - ~~That is, all optimal Θ with no idle time have inversions~~

Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in Θ so i before j but $d_i > d_j$

Assume WLOG that j follows i immediately in Θ

Construct $\bar{\Theta}$ by swapping (exchanging) i and j:



$\max(\ell_i, \ell_j) = \ell_j$

$\bar{\ell}_i, \bar{\ell}_j \le \ell_j$ so $\max(\bar{\ell}_i, \bar{\ell}_j) \le \max(\ell_i, \ell_j)$

$$L = \max_k \ell_k = \max\left(\max_{\substack{k \neq i \\ k \neq j}} \ell_k, \max(\ell_i, \ell_j)\right) \ge \max\left(\max_{\substack{k \neq i \\ k \neq j}} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right) = \bar{L}$$

So $\bar{L} \le L$. Also, $\bar{L} \ge L$ since Θ is optimal. Hence $L = \bar{L}$ and $\bar{\Theta}$ is optimal too and has fewer inversions than Θ $\Rightarrow\Leftarrow$

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

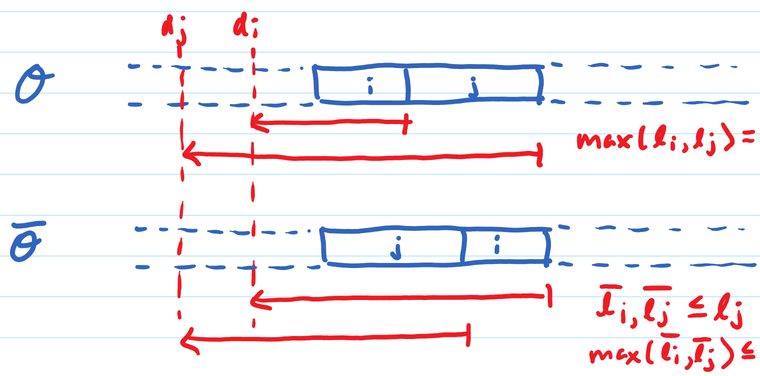
~~Suppose not - that is, all optimal Θ with no idle time have inversions~~

Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in Θ so i before j but $d_i > d_j$

Assume WLOG that j follows i immediately in Θ

Construct $\bar{\Theta}$ by swapping (exchanging) i and j:



$\max(\ell_i, \ell_j) = \ell_j$

$\bar{\ell}_i, \bar{\ell}_j \le \ell_j$ so $\max(\bar{\ell}_i, \bar{\ell}_j) \le \max(\ell_i, \ell_j)$

$$L = \max_k \ell_k$$
$$= \max\Big(\max_{\substack{k \ne i \\ k \ne j}} \ell_k, \max(\ell_i, \ell_j)\Big)$$
$$\ge \max\Big(\max_{\substack{k \ne i \\ k \ne j}} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\Big)$$
$$= \bar{L}$$

So $\bar{L} \le L$. Also, $\bar{L} \ge L$ since Θ is optimal. Hence $L = \bar{L}$ and $\bar{\Theta}$ is optimal too and has fewer inversions than Θ $\Rightarrow\Leftarrow$

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

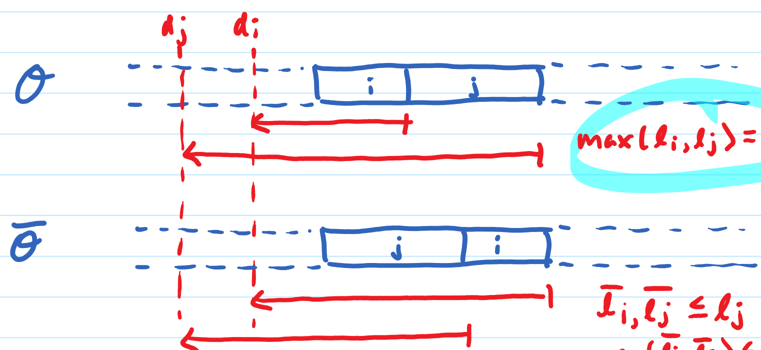
~~Suppose not - that is, all optimal $\mathcal{O}$ with no idle time have inversions~~

Pick optimal $\mathcal{O}$ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in $\mathcal{O}$ so i before j but $d_i > d_j$

Assume WLOG that j follows i immediately in $\mathcal{O}$

Construct $\bar{\mathcal{O}}$ by swapping (exchanging) i and j:



$\max(\ell_i, \ell_j) = \ell_j$

$\bar{\ell}_i, \bar{\ell}_j \le \ell_j$ so $\max(\bar{\ell}_i, \bar{\ell}_j) \le \max(\ell_i, \ell_j)$

$$L = \max_k \ell_k = \max\left(\max_{\substack{k \ne i \\ k \ne j}} \ell_k, \max(\ell_i, \ell_j)\right) \ge \max\left(\max_{\substack{k \ne i \\ k \ne j}} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right) = \bar{L}$$

$$\ell_j = f(j) \dot{-} d_j$$ def lateness

$$\ell_i = f(i) \dot{-} d_i \le f(j) \dot{-} d_i \le f(j) \dot{-} d_j = \ell_j$$

def lateness; $f(j) > f(i)$ since j after i; $d_i > d_j$ since inverted; def lateness

so $\max(\ell_i, \ell_j) = \ell_j$ $\qquad$ $\ell_i \le \ell_j$; def max

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

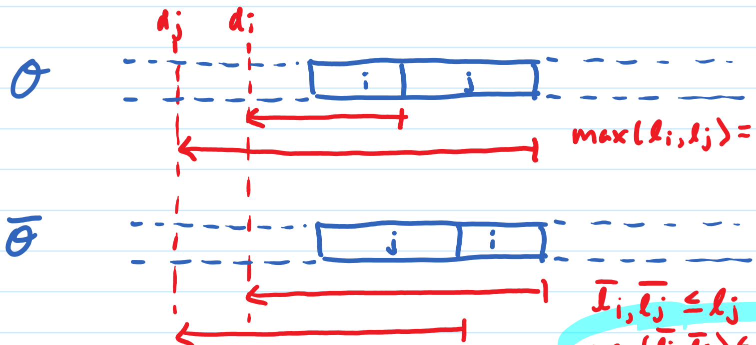
~~Suppose not - that is, all optimal θ with no idle time have inversions~~

Pick optimal θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in θ so i before j but $d_i > d_j$

Assume WLOG that j follows i immediately in θ

Construct $\bar{\theta}$ by swapping (exchanging) i and j:



$\max(\ell_i, \ell_j) = \ell_j$

$\bar{\ell}_i, \bar{\ell}_j \le \ell_j$ so $\max(\bar{\ell}_i, \bar{\ell}_j) \le \max(\ell_i, \ell_j)$

$$L = \max_k \ell_k$$
$$= \max\Big(\max_{\substack{k \ne i \\ k \ne j}} \ell_k, \max(\ell_i, \ell_j)\Big)$$
$$\ge \max\Big(\max_{\substack{k \ne i \\ k \ne j}} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\Big)$$
$$= \bar{L}$$

lateness of j went down since it was moved up

$$\underbrace{\bar{\ell}_j = \bar{f}(j) \dot{-} d_j}_{\text{def lateness}} \underbrace{\le}_{\substack{\bar{f}(j) \le f(j) \\ \text{since } j \text{ moved up}}} \underbrace{f(j) \dot{-} d_j = \ell_j}_{\text{def lateness}} \quad \text{so } \bar{\ell}_j \le \ell_j$$

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

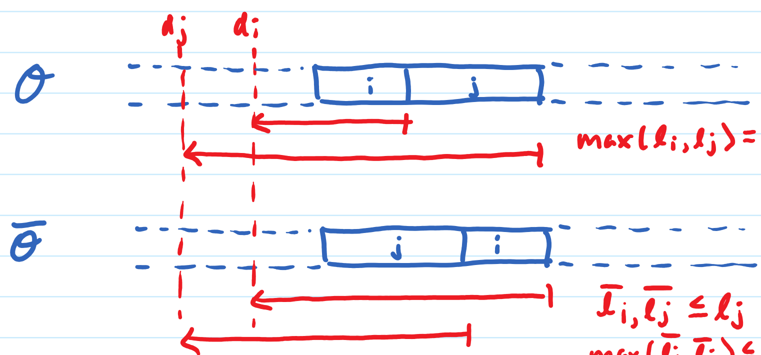
3) There is an optimal schedule with no idle time and no inversions

~~Suppose not - that is, all optimal Θ with no idle time have inversions~~

Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in Θ so i before j but $d_i > d_j$
Assume WLOG that j follows i immediately in Θ

Construct $\overline{\Theta}$ by swapping (exchanging) i and j:



$\max(\ell_i, \ell_j) = \ell_j$

$\overline{\ell}_i, \overline{\ell}_j \le \ell_j$ so $\max(\overline{\ell}_i, \overline{\ell}_j) \le \max(\ell_i, \ell_j)$

$$L = \max_k \ell_k = \max\Big(\max_{\substack{k \ne i \\ k \ne j}} \ell_k, \max(\ell_i, \ell_j)\Big) \ge \max\Big(\max_{\substack{k \ne i \\ k \ne j}} \overline{\ell}_k, \max(\overline{\ell}_i, \overline{\ell}_j)\Big) = \overline{L}$$

lateness of j went down since it was moved up

$$\overline{\ell}_j = \overline{f}(j) \dot{-} d_j \le f(j) \dot{-} d_j = \ell_j \quad \text{so } \overline{\ell}_j \le \ell_j$$

$f(j) = \sum_{x \in X} t_x + t_i + t_j = \sum_{x \in X} t_x + t_j + t_i = \overline{f}(i)$

new lateness of i not worse than lateness of j

$$\overline{\ell}_i \overset{\text{def lateness}}{=} \overline{f}(i) \dot{-} d_i \overset{\overline{f}(i) = f(j)}{=} f(j) \dot{-} d_i \overset{d_i > d_j}{\le} f(j) \dot{-} d_j \overset{\text{def lateness}}{=} \ell_j$$

so $\overline{\ell}_i \le \ell_j$

Minimizing Maximum Lateness

1) There is an optimal solution with no idle time

2) All schedules with no idle time and no inversions have the same maximum lateness

3) There is an optimal schedule with no idle time and no inversions

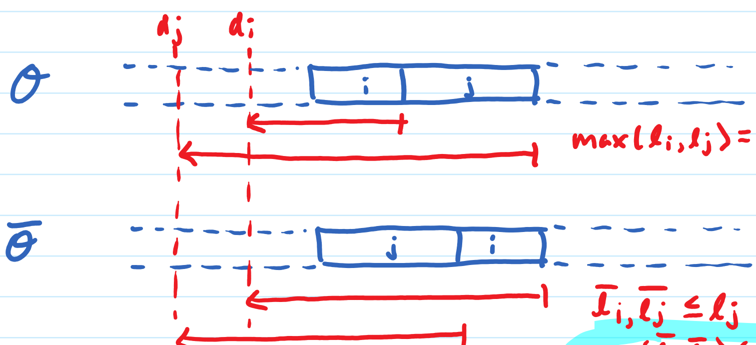
~~Suppose not - that is, all optimal Θ with no idle time have inversions~~

Pick optimal Θ with fewest inversions and no idle time (well-ordering) (1)

Let i, j be an inversion in Θ so i before j but $d_i > d_j$

Assume WLOG that j follows i immediately in Θ

Construct $\bar{\Theta}$ by swapping (exchanging) i and j:



$\max(\ell_i, \ell_j) = \ell_j$

$\bar{\ell}_i, \bar{\ell}_j \le \ell_j$ so $\max(\bar{\ell}_i, \bar{\ell}_j) \le \max(\ell_i, \ell_j)$

$$L = \max_k \ell_k = \max\left(\max_{\substack{k \ne i \\ k \ne j}} \ell_k, \max(\ell_i, \ell_j)\right) \ge \max\left(\max_{\substack{k \ne i \\ k \ne j}} \bar{\ell}_k, \max(\bar{\ell}_i, \bar{\ell}_j)\right) = \bar{L}$$

lateness of j went down since it was moved up

$$\bar{\ell}_j = \bar{f}(j) \dot{-} d_j \le f(j) \dot{-} d_j = \ell_j \quad \text{so } \bar{\ell}_j \le \ell_j$$

$f(j) = \sum_{x \in X} t_x + t_i + t_j = \sum_{x \in X} t_x + t_j + t_i = \bar{f}(i)$

def lateness; $\bar{f}(i) = f(j)$; $d_i > d_j$; def lateness

new lateness of i not worse than lateness of j

$$\bar{\ell}_i = \bar{f}(i) \dot{-} d_i = f(j) \dot{-} d_i \le f(j) \dot{-} d_j = \ell_j$$

so $\bar{\ell}_i \le \ell_j$

$$\max(\bar{\ell}_i, \bar{\ell}_j) \le \max(\ell_j, \ell_j) = \ell_j = \max(\ell_i, \ell_j)$$