Well-Ordering Principle

$\mathbb{N} = \{0, 1, 2, \ldots\}$

Mathematical Induction: Let $P(n)$ be a predicate on natural numbers

basis

If $P(0)$ is true and $\forall k \in \mathbb{N}$ s.t. $k > 0$, $P(k-1) \to P(k)$

induction step

then $P(n)$ is true for all $n \in \mathbb{N}$

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THM: $\underbrace{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}_{P(n)}$

Proof (induction on n): Basis: $P(0)$ says $\underbrace{\sum_{i=1}^{0} i = \frac{0\cdot(0+1)}{2}}_{P(0)}$, which is true because

$\sum_{i=1}^{0} i = 0$ since the sum is empty,

$\frac{0\cdot(0+1)}{2} = 0$ by arithmetic, and $0 = 0$

Induction: Suppose $k \in \mathbb{N}$, $k > 0$, and $P(k-1)$, which is $\sum_{i=1}^{k-1} i = \frac{(k-1)\cdot k}{2}$, is true.

Then $\sum_{i=1}^{k} i = \sum_{i=1}^{k-1} i + k = \frac{(k-1)\cdot k}{2} + k = \frac{(k-1)\cdot k}{2} + \frac{2k}{2}$

$= \frac{k(k+1)}{2}$

So $P(k)$ is true

$\therefore$ $\forall k \in \mathbb{N}$ s.t. $k > 0$, $P(k-1) \to P(k)$

$\therefore$ By induction, $P(n)$ is true for all $n \in \mathbb{N}$

Well-Ordering Principle

Well-Ordering Principle: Every non-empty subset S of $\mathbb{N}$ has a smallest element.

$x_0 > x_1 > x_2 > \cdots >$

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THM: $\underbrace{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}_{P(n)}$

Proof: Basis: $P(0)$ says $\sum_{i=1}^{0} i = \frac{0\cdot(0+1)}{2}$, which is true because

$\sum_{i=1}^{0} i = 0$ and $\frac{0\cdot(0+1)}{2} = 0$ and $0=0$

Induction: Suppose $k \in \mathbb{N}$, $k > 0$, and $P(k-1)$, which is $\sum_{i=1}^{k-1} i = \frac{(k-1)\cdot k}{2}$, is true.

Then $\sum_{i=1}^{k} i = \sum_{i=1}^{k-1} i + k = \frac{(k-1)\cdot k}{2} + k = \frac{(k-1)\cdot k}{2} + \frac{2k}{2}$

$= \frac{k(k+1)}{2}$

So $P(k)$ is true

$\therefore$ $\forall k \in \mathbb{N}$ s.t. $k > 0$, $P(k-1) \rightarrow P(k)$

Suppose it is not the case that $\forall n \in \mathbb{N}$, $P(n)$. Let $S = \{n \mid P(n) \text{ is false}\}$.

Then $S \neq \emptyset$ and so has a smallest element, call it k.
$k \neq 0$ and so $k > 0$ since $P(0)$ is true and so $0 \notin S$
$k-1 \notin S$ since k is the smallest in S, so $P(k-1)$ is true
So $P(k)$ is true. So $k \notin S$ $\Rightarrow\Leftarrow$

$\therefore$ By contradiction, $P(n)$ is true for all n

Well-Ordering Principle

Well-Ordering Principle: Every non-empty subset S of $\mathbb{N}$ has a smallest element.

Well-Ordering $\iff$ Mathematical Induction

$\Rightarrow$: Suppose the well-ordering principle hods but induction doesn't – there is some predicate $P(n)$ such that $P(0)$
and $\forall k \in \mathbb{N}$ s.t. $k > 0$, $P(k-1) \to P(k)$
but it is not the case that $\forall k \in \mathbb{N}, P(k)$

Let $S = \{ n \mid P(n) \text{ is false} \}$. Then S is non-empty
Let k be the smallest element of S. Then $k > 0$ since $P(0)$ and so $0 \notin S$
$k-1 \notin S$ since k is the smallest in S, so $P(k-1)$ is true.
So $P(k)$ is true $\Rightarrow\Leftarrow$ By contradiction, no such P can exist.

Well-Ordering Principle

Well-Ordering Principle: Every non-empty subset S of $\mathbb{N}$ has a smallest element.

Well-Ordering $\Longleftrightarrow$ Mathematical Induction

$\Leftarrow$: Suppose mathematical induction holds, but there is some non-empty $S \subseteq \mathbb{N}$ with no minimum element.

Choose any element $x_0 \in S$.

For $k > 0$, choose x_k to be any element in S smaller than x_{k-1}.

Now for all n, $x_n \le x_0 - n$ and $x_n \in S$ by mathematical induction

$x_0, x_1, x_2, \ldots$ smaller and smaller and all in S, so all in $\mathbb{N}$

Basis: $x_0 = x_0 - 0$ and $x_0 \in S$ by choice

Induction Step: Suppose $k > 0$ and $x_{k-1} \le x_0 - (k-1)$

By choice of x_k, $x_k < x_{k-1}$ and $x_k \in S$

so $x_k \le x_{k-1} - 1$

and $x_k \le x_0 - (k-1) - 1 = x_0 - k$

So $x_{x_0+1} \in S$ and $x_{x_0+1} \le x_0 - (x_0+1) = -1$. So $x_{x_0+1} \notin S$ $\Rightarrow\Leftarrow$

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$x_0, x_1, x_2, \ldots\ldots$ $\longrightarrow$ smaller and smaller and all in S, so all in $\mathbb{N}$

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So $x_{x_0+1} \in S$ and $x_{x_0+1} \le x_0 - (x_0+1) = -1$. So $x_{x_0+1} \notin S$ $\Rightarrow\Leftarrow$