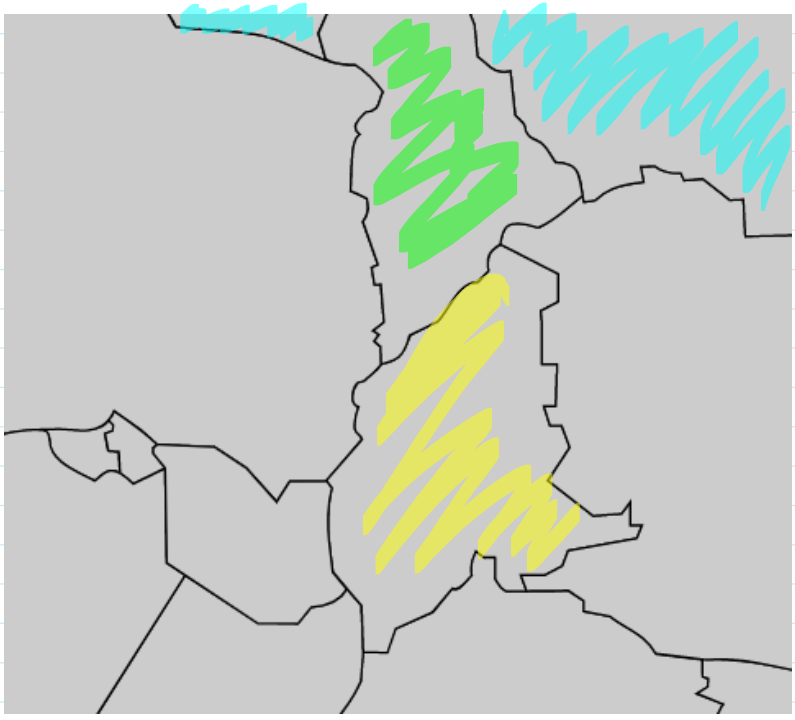
Graph Coloring



Map coloring :

assign colors to countries so neighboring countries are different colors

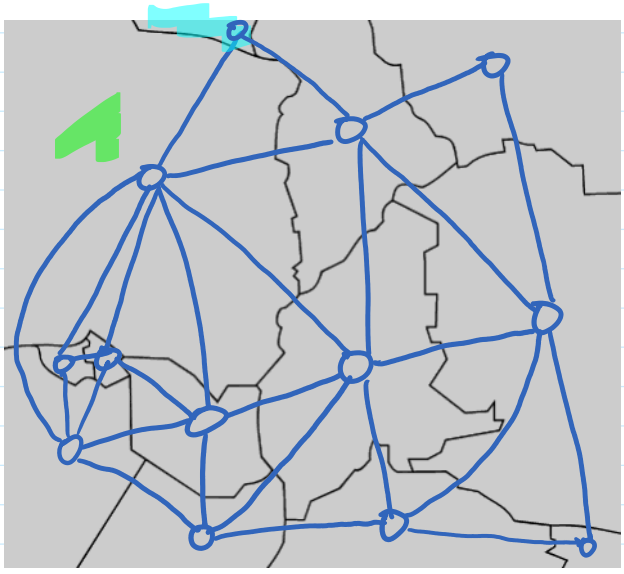
for planar graph 4 colors suffice

no edges cross

planar

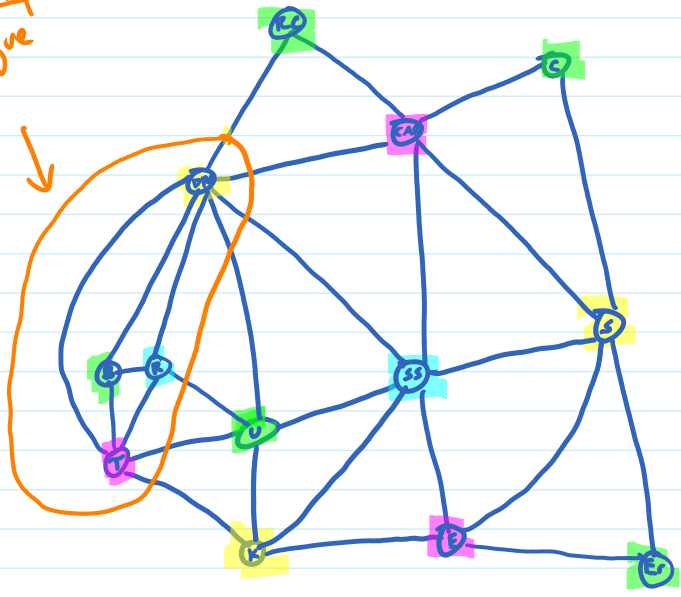
non planar

Graph coloring : Given undirected G and positive k, assign $\{1, \ldots, k\}$ ("colors") to vertices so that for every edge (u,v), $c(u) \neq c(v)$



vertices all connected to each other

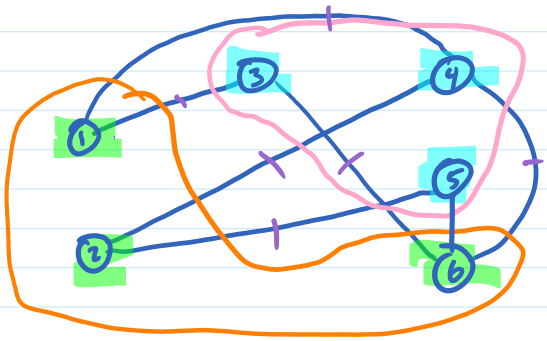
4-clique



Radio spectrum assignment

Assign one of a finite set of radio frequencies to stations so

incompatible stations are on different frequencies

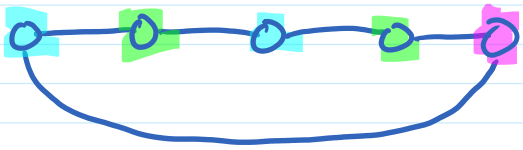


can we assign 2 frequencies?
is this graph 2-colorable?
is this graph bipartite?

710 1090

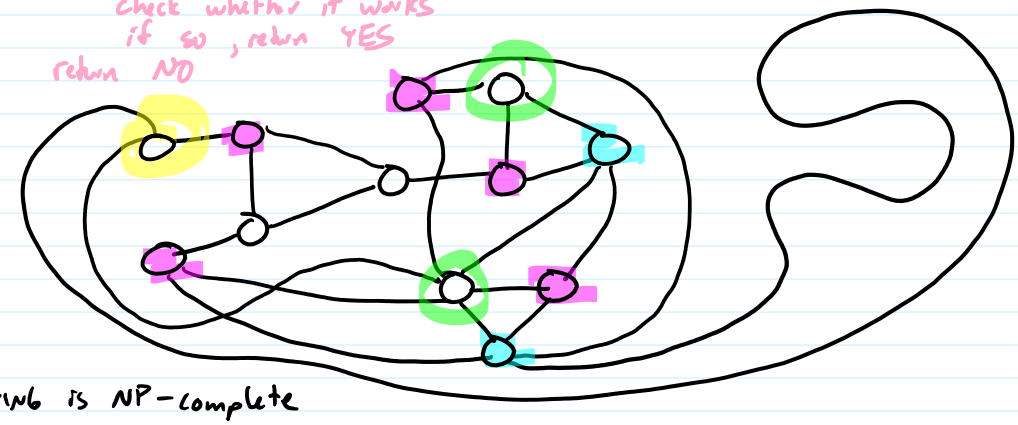
use breadth-first search

2-COLORABLE $\in P$



3-COLORING

Brute force: for each possible 3-coloring check whether it works if so, return YES
return NO ← 3^n possible 3-colorings



Prove: 3-COLORING is NP-complete

1) 3-COLORING $\in$ NP verification alg takes G, c and verifies c is a valid 3-coloring

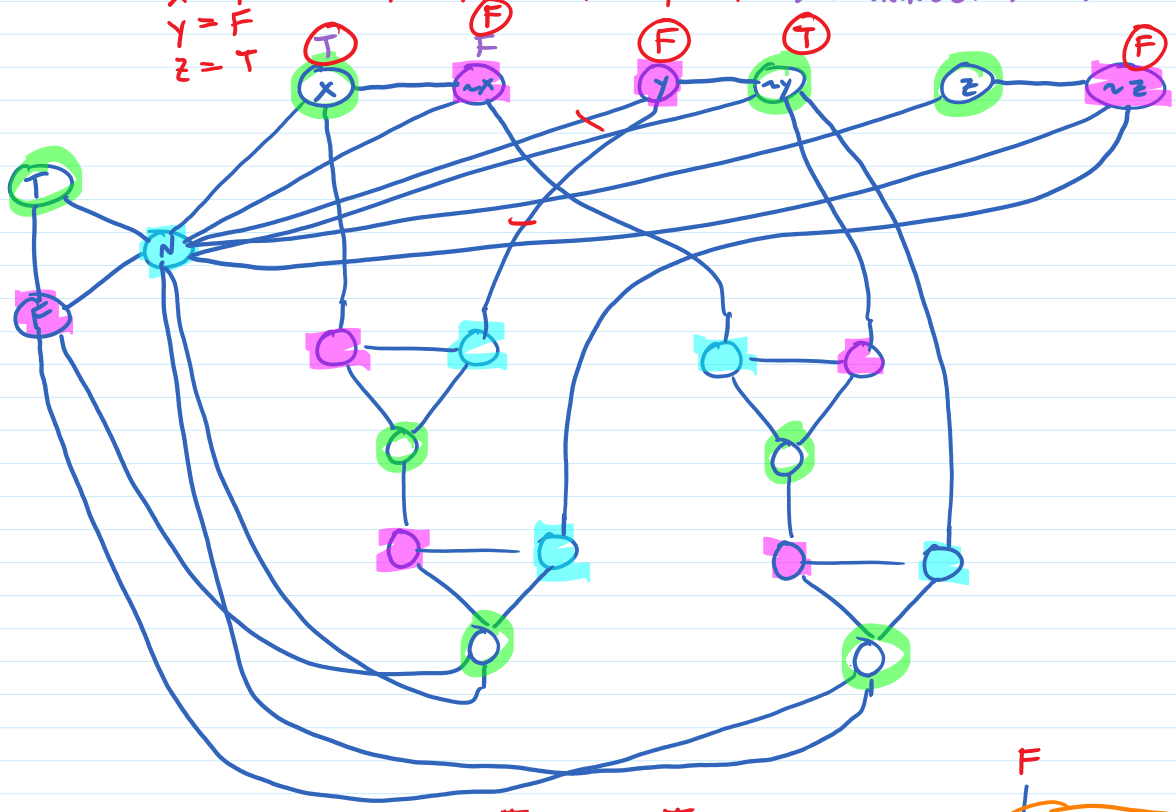
2) 3-SAT $\leq_P$ 3-COLORING [Goal: given φ, create G in poly-time s.t. φ is satisfiable $\updownarrow$ G is 3-colorable]

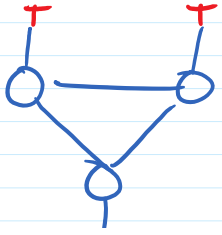
3-SAT(φ)
poly 1) create G
poly 2) return 3-COLORING(G)

3-COLORING $\in$ NP → 3-SAT $\in$ P

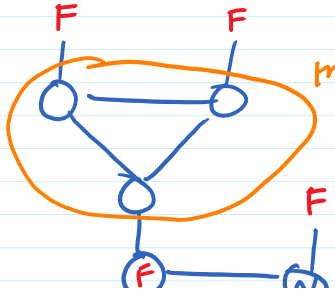
$\varphi = (x \vee y \vee \sim z) \wedge (\sim x \vee \sim y \vee \sim y)$
T F F F T T

$x = T$
$y = F$
$z = T$

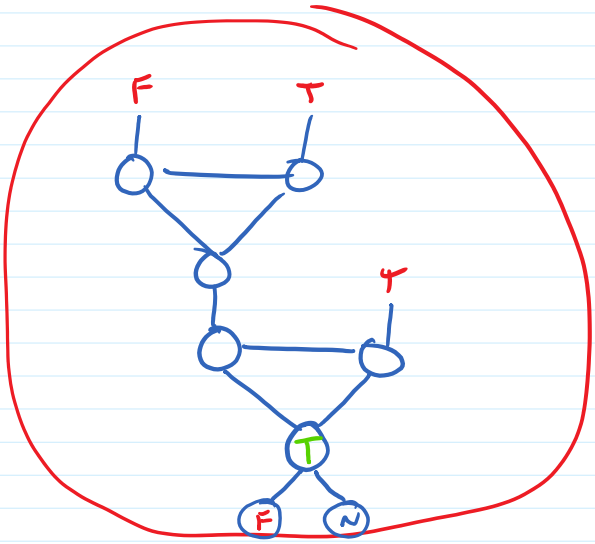


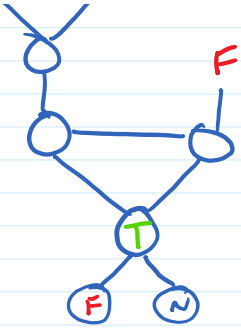


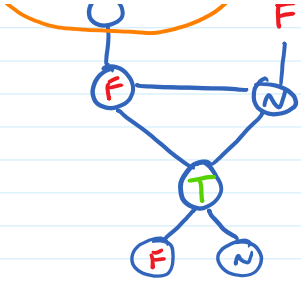
widget is 3-colorable $\updownarrow$ at least 1 of 3 inputs



at least 1 of 3 inputs is connected to T







Non-Decision Problems

$c: V \to \{R, G, B, NIL\}$ $\quad c(v)=NIL \to c'(v)=c(v)$

EXTEND-3-COLOR: Given G and a partial coloring c, can c be extended to a valid 3-coloring c'?

NP-complete (reduce from 3-COLORING w/ empty partial coloring)

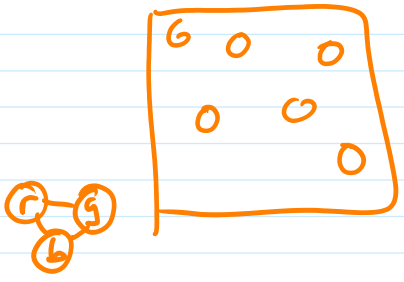
FIND-3-COLOR $\in_P$ EXTEND-3-COLOR (and so EXTEND-3-COLOR $\in P \to$ FIND-3-COLOR $\in P$)

search

```
FIND-3-COLOR(G)
  c(v) ← NIL  for all v
  if EXTEND-3-COLOR(G, c) = NO return NIL
  for each v                                   INV: EXT-3-COL(G,c) = YES
     let c_R ← c with c(v) = R
         c_G ← c with c(v) = G
         c_B ← c w/ c(v) = B
     c ← c_i s.t. EXT-3-COL(G, c_i) = YES      n+1 calls to EXT-3-COL
  return c
```

FIND-3-COLOR $\leq_P$ 3-COLORING
(If 3-COLORING $\in P$ then FIND-3-COLOR $\in P$)

```
FIND-3-COLORING(G)

  if 3-COLOR(G) = NO then return NIL

  c[v] ← NIL  for all v
  G' ← G with Δ r, g, b added

  for each vertex v
      G_R ← G' with edges (v,g) (v,b) added
      G_G ← G' w/       (v,r) (v,b) added
      G_B ← G' w/       (v,r) (v,g) added

      find i s.t. 3-COLORING(G_i) = YES
      c[v] ← i
      G' ← G_i

  return c
```



INV: G' is 3-colorable
c is a partial 3-coloring of G'
any 3-coloring of G' is a 3-coloring of G
$|\{v : c(v) \neq NIL\}|$ = # of iterations