
$a$ is willing to work with $p$ and $p$ is willing to work with a

Maximum Bipartite Matching: Given a bipartite graph, find a matching of maximum size

$$
\begin{aligned}
& x_{n} y=\varnothing \quad X, y \neq \varnothing
\end{aligned}
$$

Bipartite Graph: $V$ can be pardtioned into $X, Y$
sit. all elves $(u, v)$ ham $u \in X$ and $v \in Y$
$u \in Y$ and $v \in X$
Matching in a graph:
subset of edges MI sod. Machinists
welles


Problem: fur b matching of maximum size
ede $(m, w)$ means $m$ will work with w and vie e voes


Bipartite Matching solved by Maximum Flow
$G^{\prime}$ : input to max flow needs
source sink
cupanty
direction


1) construct $G^{\prime}$ as above
2) find max flow $f$ of $G^{\prime}$
3) out put $M=\{(x, y) \in G$ sit. $f(x, y)=1\}$


$$
\begin{aligned}
M=\{ & (A, w),(C, 4) \\
& (B, Z),(D, X)\}
\end{aligned}
$$

Lemma 1: There is an integes-valued flow $f$ in $G^{\prime}$ with $v(f)=k$
There is a matching $M$ in $G$ with $|M|=k$

Lemma 2: For directed graph 6 with integer capacities, then there is a max flow that has integer capacities.


THM: For bipartite $G$, max flow $f$ in $G^{\prime}$ gives max matching
Proof: Let $f$ be max flow, $i n$ be corresponding matching
$f$ is integer - valved
$L 2$

$$
\begin{equation*}
|m|=v(f) \tag{41}
\end{equation*}
$$

Suppose $M$ not maximum: $M^{\prime}$ is matching with $\left|M^{\prime}\right|>|M|$
Then there is corresponding flow $f^{\prime}$ with

$$
\text { So } f \text { is not max A ow } \Rightarrow=
$$

$\therefore$ So $M$ is maximum

DEF: st cut is

THM : Let $f$ be a flow, $(A, B)$ be an st cut.
Proof:

$$
\begin{aligned}
v(f) & =f^{\text {out }}(s) \\
& =f^{\text {out }}(s)-f^{\ln }(s) \\
f^{\circ o t}(v) & =f^{\prime n}(v)=0 \text { for all } v \in A-\{s\} \\
v(f) & =\sum_{v \in A} f^{\text {out }}(v)-f^{\text {in }}(v) \\
& =\sum_{v \in A} \sum_{(v, x) \in E} f(v, x)-\sum_{(u, v) \in E} f(u, v) \\
& =\sum_{v \in A} \sum_{\substack{(v, x) \in E \\
x \& A}} f(v, x)-\sum_{(u, v) \in E} f(u, v) \\
& =f^{\text {out }}(A)-f^{\text {in }}(A)
\end{aligned}
$$

Lemma 1: There is an integer -valued flow $f$ in $b^{\prime}$ with $v(f)=k$
There is a matching $M$ in $G$ with $|M|=K$
Proof : $\Rightarrow$ Construct $M=\{(x, y) \mid x \in X, y \in Y, f(x, y)=1\}$
$M$ is a mate hing in $G$

$$
\begin{aligned}
& (x, y) \in M \rightarrow(x, y) \in G \\
& \text { can't have }\left(x, y_{1}\right),\left(x, y_{2}\right) \in M, y, \not y_{2} \\
& \text { cannot hin k }\left(x_{1}, y\right),\left(x_{2}, y\right) \in M, x_{1} \not x_{2}
\end{aligned}
$$

Define st at $A=X \cup\{s\}, B=Y \cup\{t\}$

$$
\begin{aligned}
v(f) & =f^{\text {out }}(A) \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in A \\
y \& A}} f(x, y) \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in x \\
y \in 4 \\
f(x, y)=1}} f(x, y) \\
& =|\{(x, y) \mid x \in x, y \in Y, f(x, y)=1\}| \\
& =|m|
\end{aligned}
$$

Bipartite or Not?

b)


d)


Bipartite Matching solved by Maximum Flow

bipartite graph 6


$$
\begin{aligned}
& O(n+m)=O(m) \downarrow^{\prime} \\
& \quad \text { capacity graph } 6^{\prime}
\end{aligned}
$$

$\alpha(m)=O(n m) \downarrow$ Ford-Fulkerson maximum flow $f$ $O(m) \downarrow$
maximum matching $M$ total: O(nm)
$G$ has same vertices as $G$, with source $s$ and sink $t$ added
and edges $\left(s, x_{i}\right)$ for all $x_{i} \in X,\left(x_{i}, y_{j}\right)$ for all $\left(x_{i}, y_{j}\right) \in E$, and $\left(y_{j}, t\right)$ for all $\left.y_{j} \in\right\}$ all edges have capacity I

$$
M=\left\{\left(x_{i}, y_{j}\right) \text { sch that } f\left(x_{i}, y_{j}\right)=1\right\}
$$


$G$ has same vertices as $G$, with source $s$ and sink $t$ added and edges $\left(s, x_{i}\right)$ for all $x_{i} \in X,\left(x_{i}, y_{j}\right)$ for all $\left(x_{i}, y_{j}\right) \in E$, and $\left(y_{j}, t\right)$ for all $y_{j} \in T$ all edges have capacity I

$$
M=\left\{\left(x_{i}, y_{j}\right) \text { such that } f\left(x_{i}, y_{j}\right)=1\right\}
$$

Bipartite Matching solved by Maximum Flow
 maximum flow $f$ $\downarrow$ maximum matching $M$
$G$ has same vertices as $G$, with source $s$ and sink $t$ added and edges $\left(s, x_{i}\right)$ for all $x_{i} \in X,\left(x_{i}, y_{j}\right)$ for all $\left(x_{i}, y_{j}\right) \in E$, and $\left(y_{j}, t\right)$ for all $y_{j} \in T$ all edges have capacity I

Bipartite Matching solved by Maximum Flow

bipartite graph 6 capacity graph $6^{\prime}$
$G$ has same vertices as $G$, with source $s$ and sink $t$ added and edges $\left(s, x_{i}\right)$ for all $x_{i} \in X$,

Bipartite Matching solved by Maximum Flow


Lemma 1: There is an integes-valued flow $f$ in $b^{\prime}$ with $v(f)=k$
There is a matching $M$ in $G$ with $|M|=k$
Leman 2: For directed graph 6 with integer capacities, then there is a max flow with integer values. Proof: the flow returned from ford-Fulkerson is integer-valved and max ilium

THM: For bipartite $G$ max flow $f$ in $G$ gives max matching
Proof: $\quad$ Let $f$ max flow
$f$ is integer-valued LEMMA $z$
Find $m$ s.t. $|m|=k \quad$ LEMMA $~ d ~\left(m=\left\{\left(x_{i}, y_{j}\right)\right.\right.$ st. $\left.\left.f\left(x_{i}, y_{j}\right)=1\right\}\right)$
Suppose\& $M$ not maximum: $M^{\prime}$ is mottling with $\left|M^{\prime}\right|>|M|$
Then there is corresponding flow $f^{\prime}$ with

$$
v\left(f^{\prime}\right)=\left|m^{\prime}\right|>|m|=v(f)
$$

Lemma $1 \uparrow$
So $f$ is not max flow
So $M$ is maximum

DEF: set cot is partition of $V$ into $A, B$ sot. $s \in A, t \in B$

THM: Let $f$ be a flow, $(A, B)$ be an set cut. Then gout $(A)-f_{11}^{\text {in }}(A)=J(f)$ $f^{\text {in }}(B)=f^{\text {out }}(B)$

Proof:

$$
\begin{aligned}
& v(f)=f \text { out }(s) \\
& =f_{\text {out }}(s)-f^{\text {in }}(s) \\
& \text { def } \\
& \text { no eds in so } f^{\operatorname{ir}}(s)=0 \\
& f^{\circ}(1 v)-f^{\prime n}(v)=0 \text { for all } v \in A-\{s\} \quad \text { conservation } \\
& v(f)=\sum_{v \in A} f^{\text {out }}(v)-f^{\text {in }}(v) \quad v \neq s \text { terms are } 0 \\
& =\sum_{v \in A}\left(\sum_{(v, x) \in E} f(v, r)-\sum_{(u, v) \in E} f(u, v)\right) \text { def foot, fin; } \sum_{v} \sum_{(v, n)}=\sum_{(v, n)} \\
& =\sum_{v \in A}\left(\sum_{\substack{(v, x) \in E \\
x \notin A}} f(v, x)-\sum_{\substack{(u, v) \in E \\
u \notin A}} f(u, v)\right) \text { eds } A \rightarrow 1 \text { cancel } \\
& =f^{\text {at }}(A)-f^{\text {in }}(A) \text { ceavrange; } d e f f^{\text {in }}(A), f^{\text {out }}(A)
\end{aligned}
$$

Lemma 1: There is an integer - valued flow $f$ in $b^{\prime}$ with $v(f)=k$
There is a matching $m$ in $G$ with $|M|=k$
Proof: $\Rightarrow$ Construct $m=\{(x, y) \mid x \in x, y \in Y, f(x, y)=1\}$
$M$ is a mate ching in $G$
$(x, y) \in M \rightarrow(x, y) \in G \quad$ no educ added wotreen $X, y$
cant have $\left(x, y_{1}\right),\left(x, y_{2}\right) \in M, y_{1} \not y_{7}$ - otherwise $f\left(x_{2} y_{1}\right)=f\left(x, y_{7}\right)=1$, so cant his $\left(x_{1}, y\right),\left(x_{2}, y\right)+M, x_{1}, x_{2}$-similar $\quad f^{\text {ant }}(x) \geq 2$ so

$$
f(3, x)=2
$$

Define sit at $A=X \cup\{s\}, B=Y \cup\{t\}$

$$
\begin{aligned}
& v(f)=f^{\text {out }}(A)-f^{\text {in }}(A)=f^{\text {out }}(A) \quad \text { proc ism consul. of } 6^{\prime} \text { allows } \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in A \\
y \in A}} f(x, y) \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in x \\
y \in Y \\
f(x, y)=1}} f(x, y) \\
& \text { def } \\
& \text { all other flows ant } O
\end{aligned}
$$

$$
\begin{aligned}
& =\mid\{(x, y) \mid x \in x, y \in\}, f(x, y)=1\} \mid \\
& =|m|
\end{aligned}
$$

Bipartite Matching solved by Maximum Flow
THM: For bipartite $G$, max flow $f$ in $G^{\prime}$ gives max matching Proof:

Let $f$ be max flow
$f$ is integer -valued LEMMA $z$
Find $m$ s.t. $|m|=k \quad$ LEmmA $\mid \downarrow\left(m=\left\{\left(x_{i}, y_{j}\right)\right.\right.$ st. $\left.\left.f\left(x_{i}, y_{j}\right)=1\right\}\right)$
Suppose $M$ not maximum: $M^{\prime}$ is mateling with $\left|M^{\prime}\right|>|M|$
Then there is corresponding flow $f^{\prime}$ with

$$
v\left(f^{\prime}\right)=\left|m^{\prime}\right|>|m|=v(f) \quad \text { LEMMA । } \uparrow
$$

So $f$ is not max flow $\Rightarrow \Leftarrow$
$\therefore$ So $M$ is maximum

Proof:

$$
\begin{aligned}
& v(f)=f \text { out }(s) \\
& =f^{\circ 0+}(s)-f^{\ln }(s) \\
& \text { no eds in so } \operatorname{fir}(s)=0 \\
& f^{\circ o t}(v)-f^{\prime n}(v)=0 \text { for all } v \in A-\{s\} \quad \text { conservation } \\
& v(f)=\sum_{v \in A} f^{o u t}(v)-f^{\text {in }}(v) \quad v \neq s \text { terms are } 0 \\
& =\sum_{v \in A}\left(\sum_{(v, x) \in E} f(v, x)-\sum_{(u, v) \in E} f(u, v)\right) \text { def fort, fin } ; \sum_{v} \sum_{(v, n)}=\sum_{(v, v)} \\
& =\sum\left(\sum f(v, x)-\sum f(u, v)\right) \text { eds } A \rightarrow A \text { cancel }
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{v \in A}\left(\sum_{\substack{(v, x) \in E \\
x \notin A}} f(v, x)-\sum_{\substack{u, v) \in E \\
u \notin A}} f(u, v)\right) \text { edges } A \rightarrow A \text { cancel } \\
& =f^{\text {out }}(A)-f^{\text {in }}(A) \quad \text { rearrange; ; def } f^{\text {in }}(A), f^{\text {out }}(A)
\end{aligned}
$$

1: There is an intege-valued flow $f$ in $G^{\prime}$ with $v(f)=k$
There is a matching $m$ in $G$ with $|M|=k$
Proof: $\Rightarrow$ Construct $M=\{(x, y) \mid x \in X, y \in Y, f(x, y)=1\}$
$M$ is a matching in $G$
$(x, y) \in M \rightarrow(x, y) \in G$
no educ added between $X, Y$
can' $f$ have $\left(x, y_{1}\right),\left(x, y_{2}\right) \in M, y_{1} \not y_{2}$ — otherwise $f\left(x_{2}, y_{1}\right)=f\left(x, y_{7}\right)=1$, so cant hin $\left(x_{1}, y\right),\left(x_{2}, y\right)+M, x_{1}, x_{2}$ similar $\quad f^{a+t}(x) \geq 2$ so $f^{\sin (x) \geq 2}$ so $f(s, x)=2$
Define sit at $A=X \cup\{s\}, B=\{\cup\{t\}$

$$
\begin{aligned}
v(f) & =f^{\text {out }}(A)-f^{\text {in }}(A)=f^{\text {out }}(A) \quad \text { poo TAm: canst. of } b^{\prime} \text { allows } \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in A \\
y \in A}} f(x, y) \quad \text { no engluts int } A \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in x \\
y \in Y \\
f(x, y)=1}} f(x, y) \quad \text { all other flows ant } O \\
& =|\{(x, y) \mid x \in x, y \in Y, f(x, y)=1\}| \\
& =|m|
\end{aligned}
$$

Bipartite Matching solved by Maximum Flow


LEmma 1: There is an integes-valued flow $f$ in $l^{\prime}$ with $v(f)=k$
There is a matching $M$ in $G$ with $|M|=k$
Lemma 2: For directed graph 6 with integer capacities, then there is a max flow with integer values.


THM : Let $f$ be a How, $(A, B)$ be an s-t cut. Then $f^{\prime \prime}(A)-f_{" 1}(A)=v(f)$ $f^{\text {ing }}(B)-f^{\text {ont }}(B)$

Proof:

$$
\begin{aligned}
v(f) & =f^{\text {out }}(s) \\
& =f^{\text {out }}(s)-f^{\ln }(s)
\end{aligned}
$$

def
no edys in so $f^{\text {ir }}(s)=0$
$f^{\circ 0+}(v)-f^{\prime \prime}(v)=0$ for all $v \in A-\{s\} \quad$ conservation

$$
v(f)=\sum_{v \in A} f^{\text {out }}(v)-f^{\text {in }}(v) \quad v \neq s \text { terms are } 0
$$

$$
=\sum_{v \in A}\left(\sum_{(v, x) \in E} f(v, x)-\sum_{(u, v) \in E} f(k, v)\right) \quad \operatorname{def} f^{a 0 t}, f^{1 n} ; \sum_{v} \sum_{(v, n)}=\sum_{(v, n)}
$$

$$
=\sum_{v \in A}\left(\sum_{\substack{(v, x) \in \in \\ x \notin A}} f(v, x)-\sum_{\substack{(u, v) \in E \\ u \notin A}} f(u, v)\right) \text { edges } A \rightarrow A \text { cancel }
$$

$$
=f^{\text {att }}(A)-f^{\text {in }}(A) \quad \text { rearrange ; def } f^{\text {in }}(A), f^{\text {out }}(A)
$$

1: There is an intege-valued flow $f$ in $G^{\prime}$ w th $v(f)=k$
There is a matching $m$ in $G$ with $|M|=k$
Proof: $\Rightarrow$ Construct $M=\{(x, y) \mid x \in X, y \in Y, f(x, y)=1\}$
$M$ is a matching in $G$
$(x, y) \in M \rightarrow(x, y) \in G$
no educ added bardeen $X, Y$
can' $f$ have $\left(x, y_{1}\right),\left(x, y_{2}\right) \in M, y_{1} \not y_{2}$ - otherwise $f\left(x_{2}, y_{1}\right)=f\left(x, y_{7}\right)=1$, so $\begin{array}{lll}\text { cant hin } & \left(x_{1}, y\right),\left(x_{2}, y\right)+M, x_{1}, x_{2} \text {-similar } \quad f^{\text {ant }}(x) \geq 2 \text { so } \\ f^{\text {in }}(x) \geq 2 & \text { so }\end{array}$ $f(s, x)=2$
Define sit at $A=X \cup\{s\}, B=\{\cup\{t\}$

$$
\begin{aligned}
v(f) & =f^{\text {out }}(A)-f^{\text {in }}(A)=f \circ{ }^{\text {int }}(A) \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in A \\
y \& A}} f(x, y) \\
& =\sum_{\substack{(x, y) \in E^{\prime} \\
x \in x \\
y \in Y \\
f(x, y)=1}} f(x, y) \\
& =|\{(x, y) \mid x \in x, y \in Y, f(x, y)=1\}| \\
& =|m|
\end{aligned}
$$

prov Tam; canst. of $6^{\prime}$ allows no ergs int $A$
all other flows ane $O$
substation
$\Leftarrow$ similar

