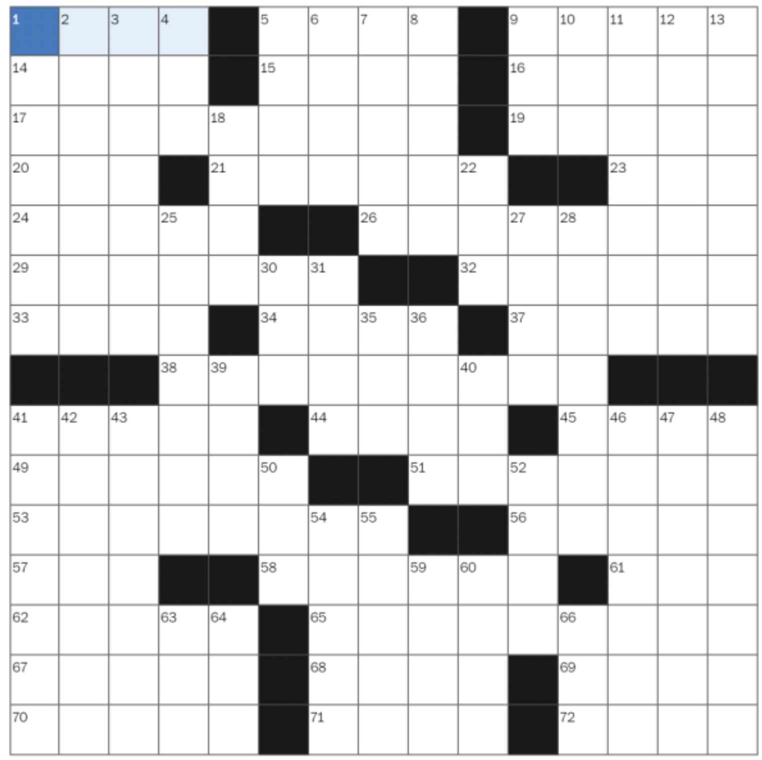
Friday, March 20, 2020 5:47 PM

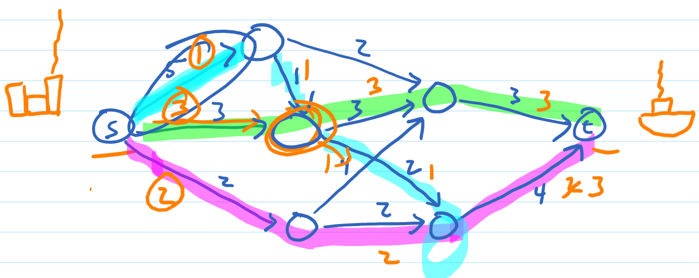


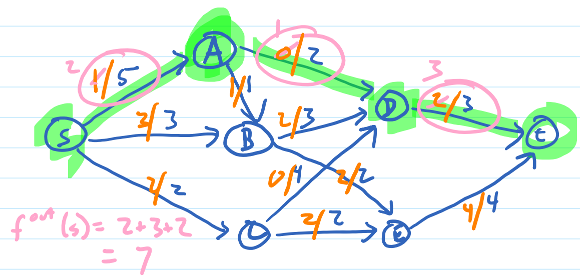
Maximum Flow

Problem: Given directed graph G with source s (no edges in) and sink t (no edges out) and capacity $c(e) > 0$ for each $e \in E$, find flow of maximum value

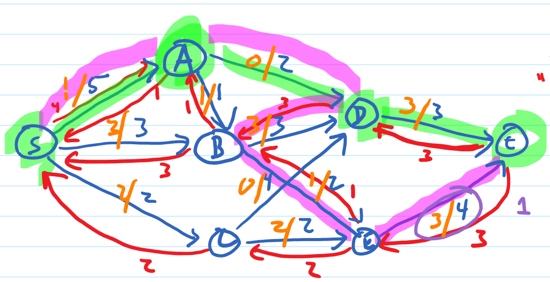
assignment of $f(u,v) \geq 0$ to each edge
such that $f(u,v) \leq c(u,v)$ for each $(u,v) \in E$ capacity
and for all $v \in V - \{s,t\}$, $\underbrace{\sum_{(u,v)\in E} f(u,v)}_{f^{in}(v)} = \underbrace{\sum_{(v,u)\in E} f(v,u)}_{f^{out}(v)}$ conservation

value of flow $v(f) = \sum_{(s,x)} f(s,x) = f^{out}(s)$



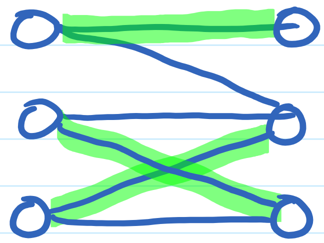


residual capacity

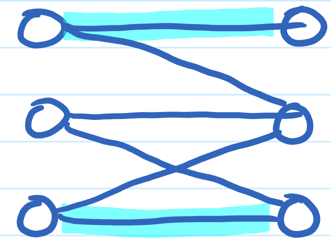


Matching or Not?

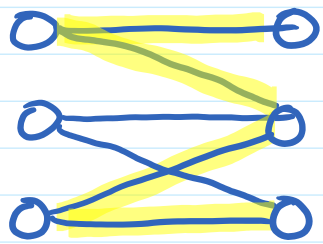
a)



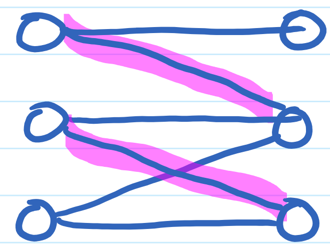
c)



b)



d)



Captions

Thursday, March 19, 2020 2:37 PM

Bipartite Graph: V can be partitioned into X and Y so that $\forall (u,v) \in E, (u \in X \text{ and } v \in Y)$ or $(u \in Y \text{ and } v \in X)$

Matching: $M \subseteq E$ such that $\forall u \in V$, if $(u, v_1) \in M$ and $(u, v_2) \in M$ then $v_1 = v_2$

$\{(a,p),(b,q),(c,t)\}$ is a matching of size 3

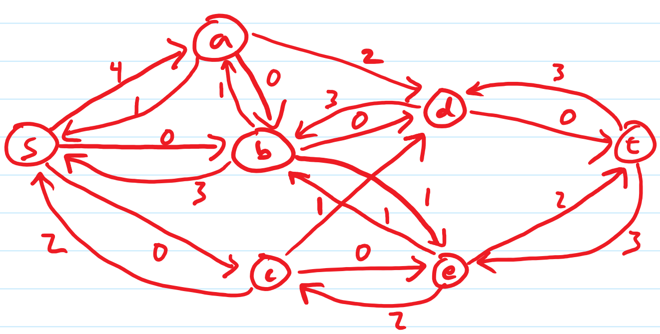
Neither $\{(a,p),(b,q),(c,t),(d,q)\}$ nor $\{(a,p),(b,q),(c,t),(d,t)\}$ is a valid matching

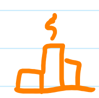
Flow : function f from edges to $\mathbb{R}$ such that

1) for each edge $(u,v) \in E$, $0 \le f(u,v) \le c(u,v)$

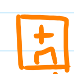
2) for each vertex $v \in V - \{s,t\}$, $\sum_{(u,v)\in E} f(u,v) = \sum_{(v,w)\in E} f(v,w)$

value of flow $= v(f) = \sum_{(s,v)\in E} f(s,v)$



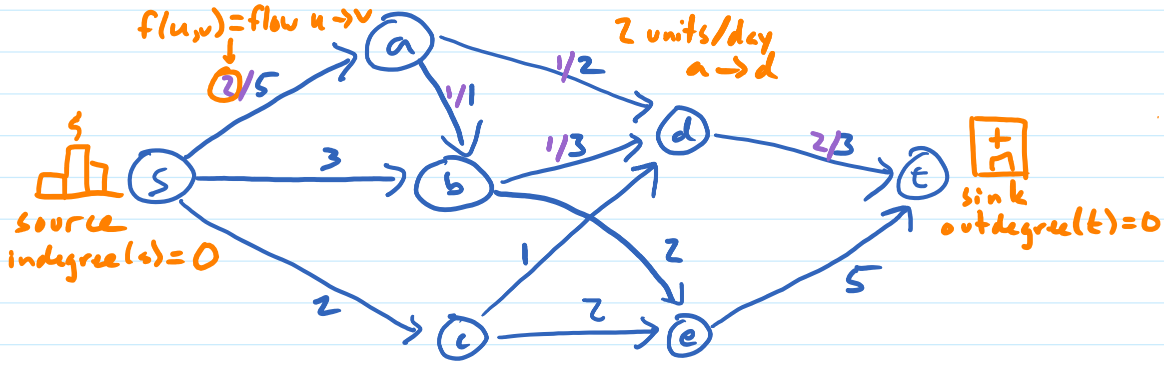


source
indegree(s) = 0



sink
outdegree(t) = 0

Maximum Flow



Flow : function f from edges to $\mathbb{R}$ such that

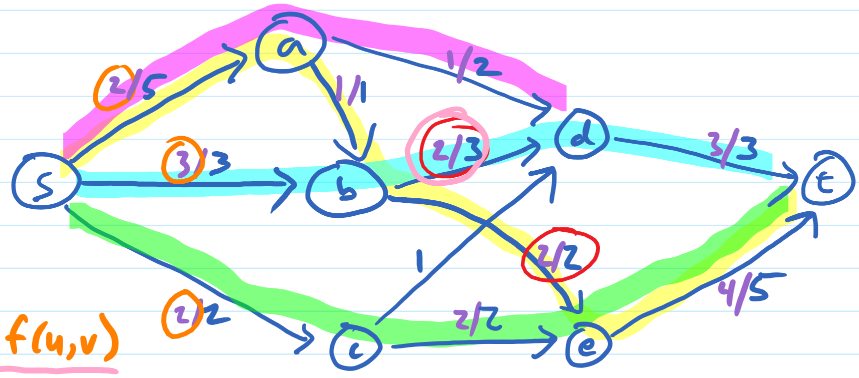
capacity 1) for each edge $(u,v) \in E$, $0 \le f(u,v) \le c(u,v)$

conservation 2) for each vertex $v \in V - \{s,t\}$, $\underbrace{\sum_{(u,v)\in E} f(u,v)}_{f^{in}(v)} = \underbrace{\sum_{(v,w)\in E} f(v,w)}_{f^{out}(v)}$

value of flow $= v(f) = \sum_{(s,v)\in E} f(s,v) = f^{out}(s)$

Maximum Flow

$v(f) = 2+3+2 = 7$



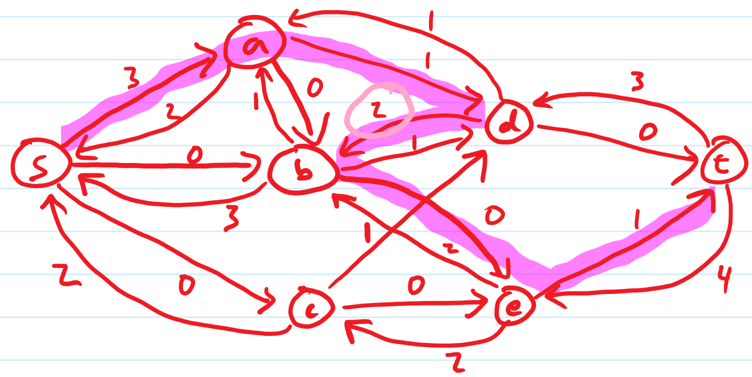
residual capacity

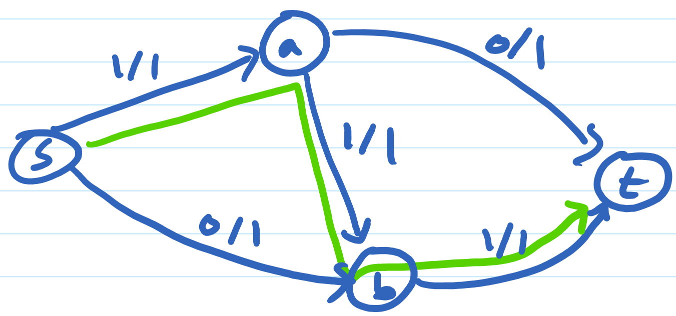
$c_r(u,v) = c(u,v) - f(u,v)$

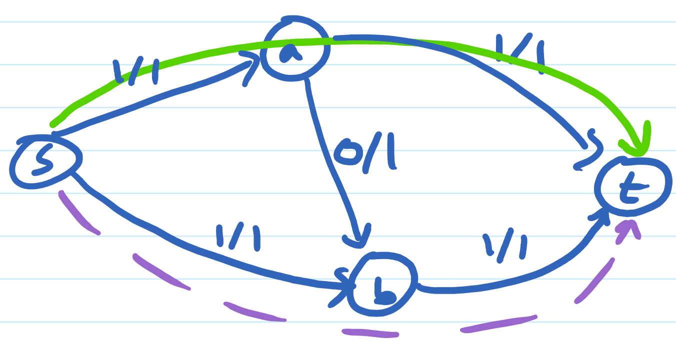
forward edges

$c_r(u,v) = f(v,u)$

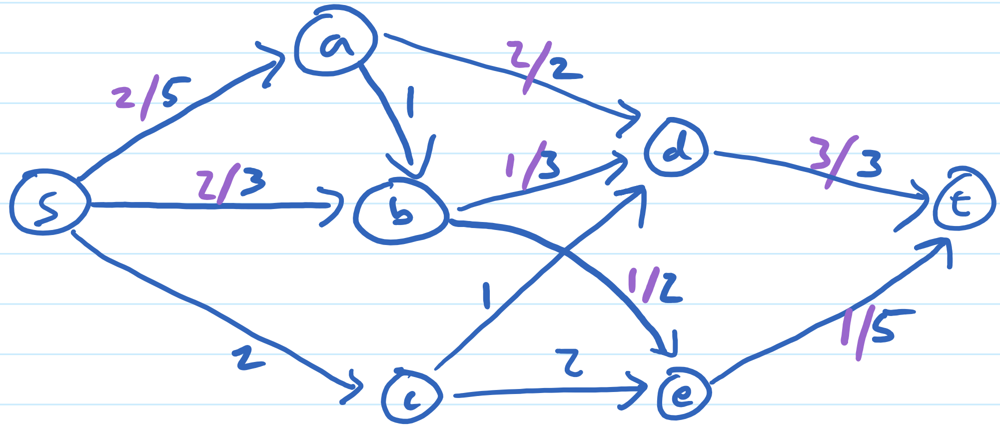
backward edges



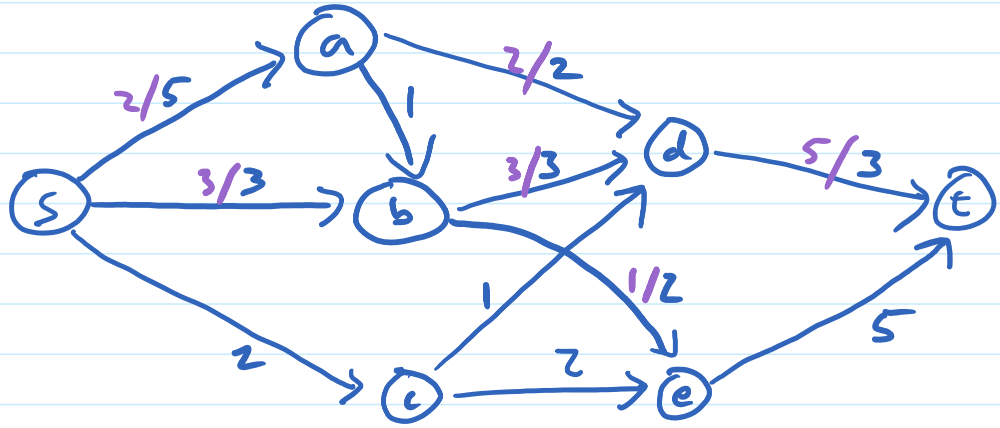




Flow In/Flow Out?



Is this a Flow?



Ford-Fulkerson

<u>MAX-FLOW-FF(G, s, t)</u>

PRECONDITIONS: s is source, t is sink
every other vertex is on some path $s \rightsquigarrow t$
all capacities are positive integers
$(u,v) \in E \rightarrow (v,u) \notin E$

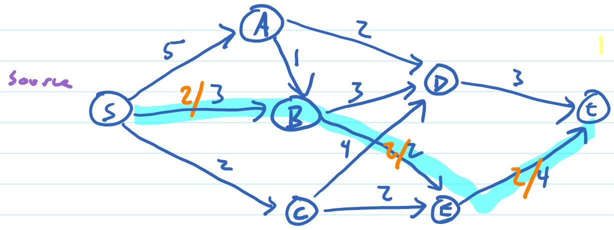
$O(C \cdot m)$ total

pseudopolynomial
(C is exponential in # bits in capacities)

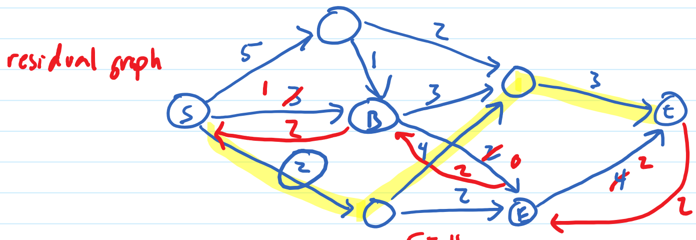
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \rightsquigarrow t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$ — C iterations where C = value of max flow
    P ← one of those paths — BFS or DFS $O(m)$
    b ← bottleneck(P) — $O(1)$ per edge in P = $O(n)$
    for $(x,y) \in P$
        if (x,y) is a forward edge
            $f(x,y) \leftarrow f(x,y) + b$
        else
            $f(y,x) \leftarrow f(y,x) - b$
        $c_r(x,y) \leftarrow c_r(x,y) - b$
        $c_r(y,x) \leftarrow c_r(y,x) + b$
    — $O(1)$ per edge = $O(n)$
    — $O(m)$ per iteration
return f

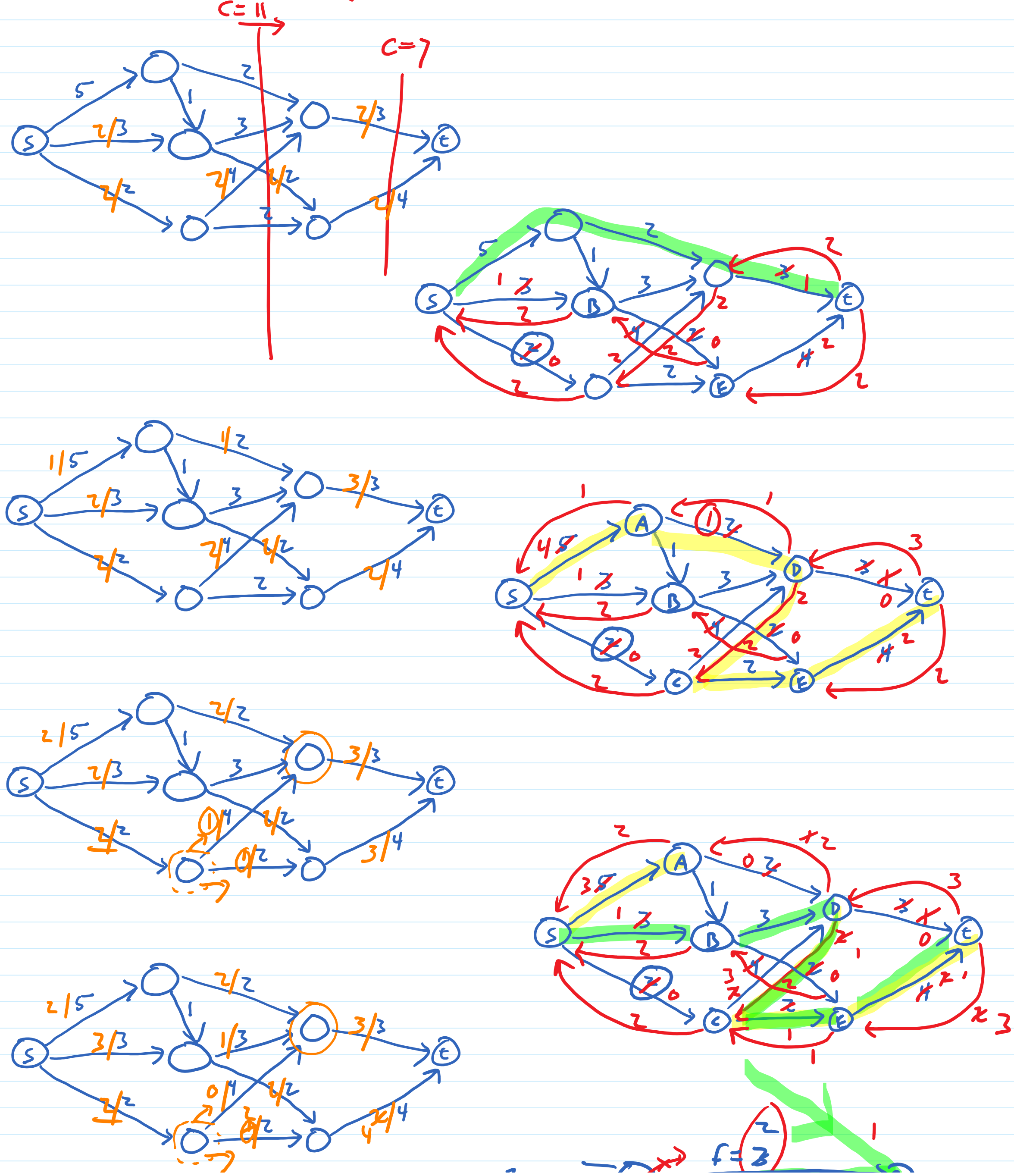
Ford-Fulkerson

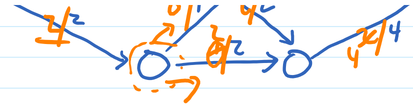


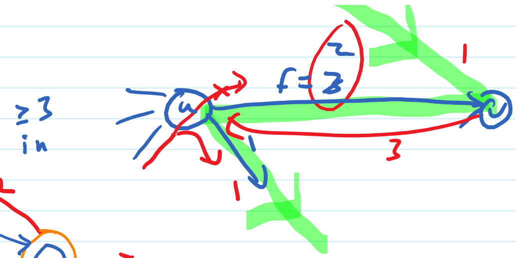
add flow of 2 along this path

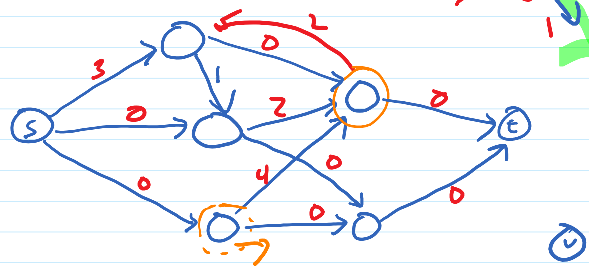


add flow of 2 along this path









MAX-FLOW-FF(G)

PRE: edge in ≤ 1 dir between u, v
capacities are pos. integer
s is source
t is sink
and all v on some path $s \rightsquigarrow t$

$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with backward edges w/ $c_r(u,v) = 0$
while there is a path P $s \rightsquigarrow t$ in G_r with all edges $(u,v) \in P$ s.t. $c_r(u,v) > 0$
  P(m) find such a path ← BFS $O(n+m) = O(m)$
  $b \leftarrow$ bottleneck(P) $O(n)$ (min $c_r(u,v)$ over edges $(u,v) \in P$)
  min of ints is an int
  for $(x,y) \in P$ — $O(n)$
    if (x,y) is forward (is in G)
      $f(x,y) \leftarrow \underbrace{f(x,y)}_{\text{int}} + \underbrace{b}_{\text{int}}$
    else
      $f(y,x) \leftarrow f(y,x) - b$
    $c_r(x,y) \leftarrow c_r(x,y) - b$
    $c_r(y,x) \leftarrow c_r(y,x) + b$
return f

C = # iterations
$C \le v(f)$
$O(n)$
$O(1)$

total $O(C \cdot m)$ pseudopolynomial
exponential in # bits in weights

INVARIANT: 1) f is flow
2) G_r is residual graph for G, f ← a) $c_r(u,v) = c(u,v) - f(u,v)$
$c_r(v,u) = f(u,v)$
for all (u,v) in G
3) f, c_r all integer-valued

Basis: (1) 0 is flow, 2) $c_r(u,v) = c(u,v)$
$= c(u,v) - f(u,v)$ since $f(u,v) = 0$
$c_r(v,u) = 0$
$= f(u,v)$ since $f(u,v) = 0$
3) $0 \in \mathbb{Z}$

Maintenance: 3)

2) for (x,y) modified in loop, if (x,y) is forward $f_{new}(x,y)$

$c_{r\,new}(x,y) =$

$=$

$=$

$c_{r\,new}(y,x) =$

if (x,y) backward

1) capacity : if (u,v) appears forward in P

$\rightarrow [f_{new}(u,v) \le c(u,v)]$

where $f_{new}(u,v) = f_{old}(u,v) + b$

$\boxed{0 < b \le c_r(u,v)} = c(u,v) - f_{old}(u,v)$

choice of b
(min c_r over edges in P)

$f_{new}(u,v) = f_{old}(u,v) + b \le c(u,v) - \cancel{f_{old}(u,v)} + \cancel{f_{old}(u,v)}$

AND $f_{old}(u,v) \ge 0$ and $b \ge 0$

so $\underbrace{f_{old}(u,v) + b}_{f_{new}(u,v)} \ge 0$

if (u,v) appears backwards in P

$0 \le b \le c_r(v,u) =$

conservation: 4 cases a) enter, leave v along forward

b) enter v along backward, leave v along forward

c)

d)

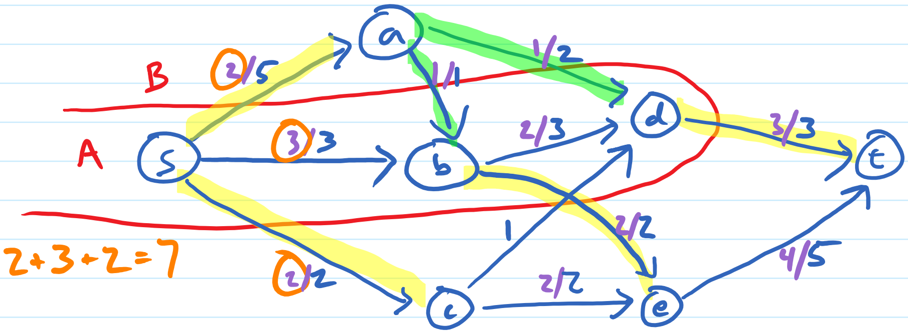
Net Flow

THM: Let f be a flow, (A,B) be an s-t cut. Then $\overbrace{f^{out}(A) - f^{in}(A)}^{\text{net flow}} = v(f)$

$f^{in}(B) \rightarrow \sum_{\substack{(u,v)\in E \\ u\in A, v\in B}} f(u,v)$ $\qquad$ $\sum_{\substack{(u,v)\in E \\ u\in B, v\in A}} f(u,v) \leftarrow f^{out}(B)$

DEF: A cut is a partition of V into A, B

An s-t cut is a cut (A,B) where $s \in A$ and $t \in B$



$v(f) = 2+3+2 = 7$

$f^{out}(A) = 2+2+2+3 = 9$

$f^{in}(A) = 1+1 = 2$

$f^{out}(A) - f^{in}(A) = 9-2 = 7$

Net Flow

THM: Let f be a flow, (A,B) be an s-t cut. Then $\overbrace{f^{out}(A) - f^{in}(A)}^{\text{net flow}} = v(f)$

DEF: A cut is a partition of V into A, B

An s-t cut is a cut (A,B) where $s \in A$ and $t \in B$

Net Flow

THM: Let f be a flow, (A,B) be an s-t cut. Then $f^{out}(A) - f^{in}(A) = v(f)$.

Proof:

$v(f) = f^{out}(s)$ — definition of $v(f)$

$= f^{out}(s) - f^{in}(s)$ — no edges (v,s) so $f^{in}(s) = 0$

$f^{out}(v) - f^{in}(v) = 0$ for all $v \in A - \{s\}$ — conservation of flow

$v(f) = \sum_{v\in A} f^{out}(v) - f^{in}(v)$ — $v \neq s$ terms are 0

$f(a_1,a_2)$ appears for $v=a_1$ and for $v=a_2$

$= \sum_{v\in A}\left(\sum_{(v,x)\in E} f(v,x) - \sum_{(u,v)\in E} f(u,v)\right)$ — definition of f^{out}, f^{in}

$= \sum_{v\in A}\left(\sum_{\substack{(v,x)\in E\\ x\notin A}} f(v,x) - \sum_{\substack{(u,v)\in E\\ u\notin A}} f(u,v)\right)$ — edges $A \to A$ cancel

$= f^{out}(A) - f^{in}(A)$ — definition of $f^{out}(A)$, $f^{in}(A)$

LEMMA 1: There is an integer-valued flow f in G' with $v(f) = k$

$\Updownarrow$

There is a matching M in G with $|M| = k$

Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x\in X, y \in Y, f(x,y) = 1\}$

M is a matching in G

$(x,y) \in M \rightarrow (x,y) \in G$ — no edges added between X, Y

can't have $(x,y_1), (x,y_2) \in M$, $y_1 \neq y_2$ — otherwise $f(x,y_1) = f(x,y_2) = 1$, so $f^{out}(x) \geq 2$ so $f^{in}(x) \geq 2$ so $f(s,x) = 2$

can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$ — similar

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$v(f) = f^{out}(A) - f^{in}(A) = f^{out}(A)$ — prev THM; const. of G' allows no edges into A

$= \sum_{\substack{(x,y)\in E'\\ x\in A\\ y\notin A}} f(x,y)$ — def

$= \sum_{\substack{(x,y)\in E'\\ x\in X\\ y\in Y\\ f(x,y)=1}} f(x,y)$ — all other flows are 0

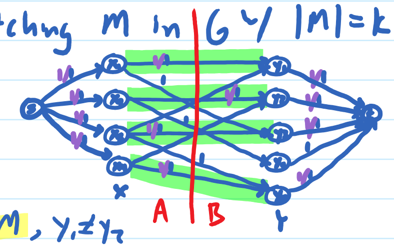
$= |\{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}|$

$= |M|$ substitution

$\Leftarrow$ similar

Net Flow

LEMMA 1: $\exists$ integer-valued flow f in G' with $v(f) = k \longleftrightarrow \exists$ matching M in G w/ $|M| = k$



Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x \in X,\ y \in Y,\ f(x,y) = 1\}$

construction of G'

M is a matching in G: $(x,y) \in M \rightarrow (x,y) \in G$

otherwise $f(x,y_1) = f(x,y_2) = 1$, so $f^{out}(x) \geq 2$ $\leftarrow$ can't have $(x,y_1),(x,y_2) \in M$, $y_1 \neq y_2$

and $f^{in}(x) \geq 2$, so $f(s,x) \geq 2 > c(s,x) = 1$ ✗

can't have $(x_1,y),(x_2,y) \in M$, $x_1 \neq x_2$ similar

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$$v(f) = f^{out}(A) - f^{in}(A) = f^{out}(A)$$ Net Flow Thm; $f^{in}(A) = 0$

$$= \sum_{\substack{(x,y)\in E' \\ x \in A \\ y \notin A}} f(x,y) = \sum_{\substack{(x,y)\in E' \\ x \in X \\ y \in Y \\ f(x,y)=1}} f(x,y) = \left|\{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}\right|$$

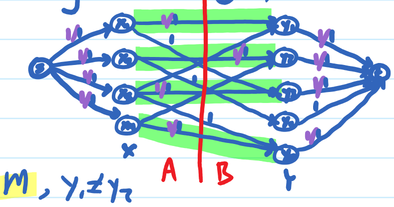
def $f^{out}(A)$

$$= |M|$$

$\Leftarrow$ similar

Net Flow

LEMMA 1: $\exists$ integer-valued flow f in G' with $v(f) = k \longleftrightarrow \exists$ matching M in G w/ $|M| = k$



Proof: $\Rightarrow$ Construct $M = \{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}$

M is a matching in G: $\overbrace{(x,y) \in M \rightarrow (x,y) \in G}^{\text{construction of } G'}$

otherwise $f(x,y_1) = f(x,y_2) = 1$, so $f^{out}(x) \geq 2$ $\leftarrow$ can't have $(x,y_1), (x,y_2) \in M$, $y_1 \neq y_2$

and $f^{in}(x) \geq 2$, so $f(s,x) \geq 2 > c(s,x) = 1$ ✗

can't have $(x_1,y), (x_2,y) \in M$, $x_1 \neq x_2$ similar

Define s-t cut $A = X \cup \{s\}$, $B = Y \cup \{t\}$

$v(f) = f^{out}(A) - f^{in}(A) = f^{out}(A)$ Net Flow Thm; $f^{in}(A) = 0$

$$= \sum_{\substack{(x,y)\in E' \\ x \in A \\ y \notin A}} f(x,y) = \sum_{\substack{(x,y)\in E' \\ x \in X \\ y \in Y \\ f(x,y)=1}} f(x,y) = |\{(x,y) \mid x \in X, y \in Y, f(x,y) = 1\}|$$

def $f^{out}(A)$ eliminate 0 terms

$= |M|$ substitution

$\Leftarrow$ similar

Net Flow

LEMMA 1: $\exists$ integer-valued flow f in G' with $v(f) = k \Longleftrightarrow \exists$ matching M in G w/ $|M| = k$

Proof: $\Leftarrow$ Construct flow f from M

Define s-t cut (A, B) s.t. $f^{out}(A) - f^{in}(A) = |M|$

$\Leftarrow$ similar

Minimum Cut

THM: If f is a flow s.t. there is no path $s \rightsquigarrow t$ in corresponding G_r, then there is a s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

$$c(A^*,B^*) = \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} c(u,v)$$

Proof: Later

COR: MAX-FLOW-FF returns a maximum flow

Proof: Let f be the flow returned by MAX-FLOW-FF — INV

There is no path $s \rightsquigarrow t$ in corresponding G_r — termination cond on while

Find s-t cut (A^*, B^*) s.t. $\boxed{v(f) = c(A^*, B^*)}$ — above THM

A^* = verts v s.t. $s \rightsquigarrow v$ in G_r

Let f' be any flow. [want $v(f') \le v(f)$]

$$\begin{aligned}
v(f') &= f'^{out}(A^*) - f'^{in}(A^*) && \text{prev THM} \\
&\le f'^{out}(A^*) && f'^{in}(A^*) \ge 0 \\
&= \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} f'(u,v) && \text{def } f'^{out}(A^*) \\
&\le \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} c(u,v) && \text{capacity constraint } (f'(u,v) \le c(u,v)) \\
&= c(A^*, B^*) = v(f) && \text{def } c(A^*,B^*)\text{; choice of } A^*
\end{aligned}$$

THM: If f is a flow s.t. there is no path $s \rightsquigarrow t$ in corresponding G_r, then there is a s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

Proof: Construct $A^* = \{ v \mid \exists \text{ path } s \rightsquigarrow v \text{ in } G_r \}$

$B^* = V - A^*$

(A^*, B^*) in an s-t cut — $s \rightsquigarrow s$; $s \not\rightsquigarrow t$ means $t \notin A^*$ so $t \in B^*$

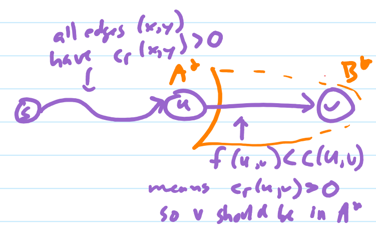
- Consider $(u,v) \in E$ where $u \in A^*$, $v \in B^*$

  Suppose $f(u,v) < c(u,v)$

  then $c_r(u,v) = c(u,v) - f(u,v) > 0$

  so there is a path $s \rightsquigarrow v$ using edges having $c_r > 0$

  $\therefore v \in A^*$ $\Rightarrow\Leftarrow$ (contradicts $v \in B^*$)



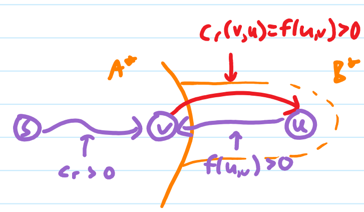
- Consider $(u,v) \in E$ where $u \in B^*$, $v \in A^*$

  Suppose $f(u,v) > 0$

  then $c_r(v,u) = f(u,v) > 0$

  so there is a path $s \rightsquigarrow u$ using edges having $c_r > 0$

  $\therefore u \in A^*$ $\Rightarrow\Leftarrow$ (contradicts $u \in B^*$)



having c_f s'O

$\therefore u \in A^* \Rightarrow\Leftarrow$ (contradicts $u \in B^*$)

$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$ conservation

$$= \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} f(u,v) - \sum_{\substack{(u,v)\in E \\ u \in B^* \\ v \in A^*}} f(u,v)$$ def f^{in}, f^{out}

$$= \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} c(u,v) - 0$$ * above

$$= c(A^*, B^*)$$ def capacity across cut

constructed as in prev thm

COR: Let f be max flow in G and (A^*, B^*) be corresponding cut. Then (A^*, B^*) is a min-capacity cut

Proof:

F-F Invariant

<u>MAX-FLOW-FF(G, s, t)</u>   PRECONDITIONS: s is source, t is sink
every other vertex is on some path $s \rightsquigarrow t$
all capacities are positive integers
$(u,v) \in E \rightarrow (v,u) \notin E$

$i \leftarrow 0$
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \rightsquigarrow t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
        $P \leftarrow$ one of those paths
        $b \leftarrow$ bottleneck(P)
        for $(x,y) \in P$
                if (x,y) is a forward edge
                        $f(x,y) \leftarrow f(x,y) + b$
                else
                        $f(y,x) \leftarrow f(y,x) - b$
                $c_r(x,y) \leftarrow c_r(x,y) - b$
                $c_r(y,x) \leftarrow c_r(y,x) + b$
        $i \leftarrow i + 1$
return f

INVARIANT: a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
forward: $c_r(u,v) = c(u,v) - f(u,v)$
backward: $c_r(u,v) = f(v,u)$

F-F Invariant: Maintenance of Capacity

$i \leftarrow 0$
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \leadsto t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
  $P \leftarrow$ one of those paths
  $b \leftarrow$ bottleneck(P)
  for $(x,y) \in P$
    if (x,y) is a forward edge
      $f(x,y) \leftarrow f(x,y) + b$
    else
      $f(y,x) \leftarrow f(y,x) - b$
    $c_r(x,y) \leftarrow c_r(x,y) - b$
    $c_r(y,x) \leftarrow c_r(y,x) + b$
  $i \leftarrow i+1$
return f

INVARIANT
a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
  forward: $c_r(u,v) = c(u,v) - f(u,v)$
  backward: $c_r(u,v) = f(v,u)$

Suppose f is a flow at the beginning of iteration i
Then $f_{old}(x,y) \leq c_r(x,y)$ for all edges (capacity)

Let (x,y) be a forward edge in P
Then $f_{new}(x,y) = f_{old}(x,y) + b$
and $b \leq c_r(x,y) = c(x,y) - f_{old}(x,y)$

So $f_{new}(x,y) \leq f_{old}(x,y) + c(x,y) - f_{old}(x,y)$
$= c(x,y)$

Also, $b > 0$, so $f_{new}(x,y) > f_{old}(x,y) \geq 0$

F-F Invariant: Maintenance of Capacity

$i \leftarrow 0$
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \rightsquigarrow t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
    $P \leftarrow$ one of those paths
    $b \leftarrow$ bottleneck(P)
    for $(x,y) \in P$
        if (x,y) is a forward edge
            $f(x,y) \leftarrow f(x,y) + b$
        else
            $f(y,x) \leftarrow f(y,x) - b$
        $c_r(x,y) \leftarrow c_r(x,y) - b$
        $c_r(y,x) \leftarrow c_r(y,x) + b$
    $i \leftarrow i+1$
return f

INVARIANT
a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
    forward: $c_r(u,v) = c(u,v) - f(u,v)$
    backward: $c_r(u,v) = f(v,u)$

Suppose f is a flow at the beginning of iteration i
Then $f_{old}(x,y) \leq c_r(x,y)$ for all edges (capacity)

Let (x,y) be a backward edge in P
Then $f_{new}(y,x) = f_{old}(y,x) - b$
and $b \leq c_r(x,y) = f_{old}(y,x)$

So $f_{new}(y,x) \geq f_{old}(y,x) - f_{old}(y,x) = 0$

Also, $b > 0$, so $f_{new}(y,x) < f_{old}(y,x) \leq c(y,x)$

F-F Invariant: Maintenance of Conservation

$i \leftarrow 0$
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \rightsquigarrow t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
    $P \leftarrow$ one of those paths
    $b \leftarrow$ bottleneck(P)
    for $(x,y) \in P$
        if (x,y) is a forward edge
            $f(x,y) \leftarrow f(x,y) + b$
        else
            $f(y,x) \leftarrow f(y,x) - b$
        $c_r(x,y) \leftarrow c_r(x,y) - b$
        $c_r(y,x) \leftarrow c_r(y,x) + b$
    $i \leftarrow i+1$
return f

INVARIANT
a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
    forward : $c_r(u,v) = c(u,v) - f(u,v)$
    backward: $c_r(u,v) = f(v,u)$

Suppose f is a flow at the beginning of iteration i
Then $f^{in}_{old}(v) = f^{out}_{old}(v)$ for all $v \in V - \{s,t\}$ (conservation)

Let v be a vertex on $P = s \rightsquigarrow x \rightarrow v \rightarrow y \rightsquigarrow t$

4 cases: a) $(x,v),(v,y)$ both forward

$f_{new}(x,v) = f_{old}(x,v) + b$
$f_{new}(v,y) = f_{old}(v,y) + b$

$f^{in}_{new}(v) = f^{in}_{old}(v) + b = f^{out}_{old}(v) + b = f^{out}_{new}(v) + b$
$f^{out}_{new}(v) = f^{out}_{old}(v) + b$

F-F Invariant: Maintenance of Conservation

$i \leftarrow 0$
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \rightsquigarrow t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
  $P \leftarrow$ one of those paths
  $b \leftarrow$ bottleneck(P)
  for $(x,y) \in P$
    if (x,y) is a forward edge
      $f(x,y) \leftarrow f(x,y) + b$
    else
      $f(y,x) \leftarrow f(y,x) - b$
    $c_r(x,y) \leftarrow c_r(x,y) - b$
    $c_r(y,x) \leftarrow c_r(y,x) + b$
  $i \leftarrow i+1$
return f

INVARIANT
a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
  forward: $c_r(u,v) = c(u,v) - f(u,v)$
  backward: $c_r(u,v) = f(v,u)$

Suppose f is a flow at the beginning of iteration i
Then $f^{in}_{old}(v) = f^{out}_{old}(v)$ for all $v \in V - \{s,t\}$ (conservation)

Let v be a vertex on $P = s \rightsquigarrow x \rightarrow v \rightarrow y \rightsquigarrow t$

4 cases: b) (x,v) forward, (v,y) backward

$f_{new}(x,v) = f_{old}(x,v) + b$
$f_{new}(y,v) = f_{old}(y,v) - b$

$f^{in}_{new}(v) = f^{in}_{old}(v) \cancel{+b} \cancel{-b} = f^{in}_{old}(v) = f^{out}_{old}(v) = f^{out}_{new}(v)$

c) backward / forward
similar

d) backward / backward
similar

F-F Invariant: Maintenance of Residuals

$i \leftarrow 0$
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \rightsquigarrow t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
    $P \leftarrow$ one of those paths
    $b \leftarrow$ bottleneck(P)
    for $(x,y) \in P$
        if (x,y) is a forward edge
            $f(x,y) \leftarrow f(x,y) + b$
        else
            $f(y,x) \leftarrow f(y,x) - b$
        $c_r(x,y) \leftarrow c_r(x,y) - b$
        $c_r(y,x) \leftarrow c_r(y,x) + b$
    $i \leftarrow i+1$
return f

INVARIANT
a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
    forward: $c_r(u,v) = c(u,v) - f(u,v)$
    backward: $c_r(u,v) = f(v,u)$

Suppose G_r is the residual graph at the start of iteration i

Let (x,y) be a forward edge in P

$c_{r,old}(x,y) = c(x,y) - f_{old}(x,y)$

$f_{new}(x,y) = f_{old}(x,y) + b$

$$\begin{aligned} c_{r,new}(x,y) = c_{r,old}(x,y) - b &= c(x,y) - f_{old}(x,y) - b \\ &= c(x,y) - (f_{old}(x,y) + b) \\ &= c(x,y) - f_{new}(x,y) \end{aligned}$$

F-F Invariant: Maintenance of Residuals

```
i ← 0
f(u,v) ← 0 for all edges (u,v)
G_r ← G with c_r(u,v) = c(u,v) and c_r(v,u) = 0 for all (u,v) ∈ E

while there is a path P s⇝t in G_r s.t. all edges (x,y) ∈ P have c_r(x,y) > 0
    P ← one of those paths
    b ← bottleneck(P)
    for (x,y) ∈ P
        if (x,y) is a forward edge
            f(x,y) ← f(x,y) + b
        else
            f(y,x) ← f(y,x) − b
        c_r(x,y) ← c_r(x,y) − b
        c_r(y,x) ← c_r(y,x) + b
    i ← i+1
return f
```

INVARIANT

a) f and c_r are integer-valued

b) $v(f) \geq i$

c) f is a flow

d) G_r is the residual graph
- forward: $c_r(u,v) = c(u,v) - f(u,v)$
- backward: $c_r(u,v) = f(v,u)$

Suppose G_r is the residual graph at the start of iteration i

Let (x,y) be a forward edge in P (similar for backward edges)

$$c_{r,old}(y,x) = f_{old}(x,y)$$

$$f_{new}(x,y) = f_{old}(x,y) + b$$

$$c_{r,new}(y,x) = c_{r,old}(y,x) + b = f_{old}(x,y) + b = f_{new}(x,y)$$

F-F Invariant: Termination

$i \leftarrow 0$
$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \rightsquigarrow t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
    $P \leftarrow$ one of those paths
    $b \leftarrow$ bottleneck(P)
    for $(x,y) \in P$
        if (x,y) is a forward edge
            $f(x,y) \leftarrow f(x,y) + b$
        else
            $f(y,x) \leftarrow f(y,x) - b$
        $c_r(x,y) \leftarrow c_r(x,y) - b$
        $c_r(y,x) \leftarrow c_r(y,x) + b$
    $i \leftarrow i + 1$
return f

INVARIANT
a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
    forward: $c_r(u,v) = c(u,v) - f(u,v)$
    backward: $c_r(u,v) = f(v,u)$

TERMINATION: $v(f) \leq \sum_{(s,v) \in E} c(s,v) = C_{bound}$

If loop is infinite then eventually $i \geq C_{bound}$
and $v(f) \geq C_{bound}$ $\Rightarrow\Leftarrow$

$\therefore$ loop is not infinite

```
i ← 0
f(u,v) ← 0 for all edges (u,v)
Gr ← G with cr(u,v) = c(u,v) and cr(v,u) = 0 for all (u,v) ∈ E

while there is a path P s⇝t in Gr s.t. all edges (x,y) ∈ P have cr(x,y) > 0
    P ← one of those paths
    b ← bottleneck(P)
    for (x,y) ∈ P
        if (x,y) is a forward edge
            f(x,y) ← f(x,y) + b
        else
            f(y,x) ← f(y,x) − b
        cr(x,y) ← cr(x,y) − b
        cr(y,x) ← cr(y,x) + b
    i ← i + 1
return f
```

INVARIANT
a) f and c_r are integer-valued
b) $v(f) \geq i$
c) f is a flow
d) G_r is the residual graph
forward: $c_r(u,v) = c(u,v) - f(u,v)$
backward: $c_r(u,v) = f(v,u)$

POSTCONDITION: the f returned is a maximum flow

f is a valid, integer-valued flow
G_r is the residual graph
there is no path $s \rightsquigarrow t$ in G_r s.t. $c_r(x,y) > 0$ for all edges on path
there is an s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

THM: If f is a flow s.t. there is no path $s \rightsquigarrow t$ in corresponding G_r, then there is an s-t cut (A^*, B^*) s.t. $v(f) = \underbrace{c(A^*, B^*)}_{\text{capacity across cut}}$ and f is max flow

$$\sum_{\substack{(a,b) \in E \\ a \in A^*, b \in B^*}} c(a,b)$$

will show $v(f') \leq v(f)$ Let f' be a flow

THM: for any flow f and s-t cut A, B

$v(f) \leq c(A,B)$

$v(f') = f'^{out}(A^*) - f'^{in}(A^*)$	Net Flow Thm
$\leq f'^{out}(A^*)$	$f'^{in}(A^*) \geq 0$
$= \sum_{\substack{(a,b) \in E \\ a \in A^*, b \in B^*}} f'(a,b)$	def f^{out}
$\leq \sum_{\substack{(a,b) \\ a \in A^*, b \in B^*}} c(a,b)$	capacity constraint
$= c(A^*, B^*)$	def capacity across cut
$= v(f)$	choice of A^*, B^*

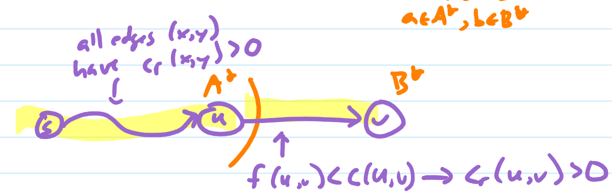
Max Flow/Min Cut

THM: If f is a flow s.t. there is no path $s \rightsquigarrow t$ in corresponding G_r, then there is an s-t cut (A^*, B^*) s.t. $v(f) = \underbrace{c(A^*, B^*)}_{\text{capacity across cut } \sum_{\substack{(a,b)\in E \\ a\in A^*,\, b\in B^*}} c(a,b)}$

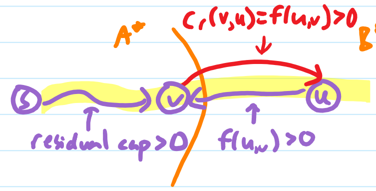
Proof: Construct $A^* = \{ v \mid \exists \text{ path } s \rightsquigarrow v \text{ in } G_r \text{ with } c_r > 0 \text{ on all edges} \}$, $B^* = V - A^*$

(A^*, B^*) is an s-t cut $s \rightsquigarrow s$; $s \not\rightsquigarrow t \to t \notin A^* \to t \in B^*$

Consider $(u,v) \in E$ where $u \in A^*$, $\underline{v \in B^*}$
  Suppose $f(u,v) < c(u,v)$. Then $v \in A^*$ $\Rightarrow\Leftarrow$
  $\therefore$ $f(u,v) = c(u,v)$ left → right maxed out



Consider $(u,v) \in E$ where $\underline{u \in B^*}$, $v \in A^*$
  Suppose $f(u,v) > 0$. Then $u \in A^*$ $\Rightarrow\Leftarrow$
  $\therefore$ $f(u,v) = 0$ right → left zeroed



$v(f) = f^{out}(A^*) - f^{in}(A^*)$ Net flow th

$$= \sum_{\substack{(u,v)\in E \\ u\in A^* \\ v \in B^*}} f(u,v) - \sum_{\substack{(u,v)\in E \\ u\in B^* \\ v \in A^*}} f(u,v) = \sum_{\substack{(u,v)\in E \\ u\in A^* \\ v \in B^*}} c(u,v) - 0 = c(A^*, B^*)$$

def f^{in}, f^{out} def capacity across cut

THM: If f is a flow s.t. there is no path $s \rightsquigarrow t$ in corresponding G_r, then there is an s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

COR: If is a maximum flow in G then a) there is no path $s \rightsquigarrow t$ with positive residual capacity on all edges

b) $c(A^*, B^*)$ is a minimum capacity cut, where
$A^* = \{ v \mid \exists \text{ path } s \rightsquigarrow v \text{ in } G_r \text{ with } c > 0 \text{ on all edges} \}$
$B^* = V - A^*$

Proof: a) if there is a path, can increase flow along it as in Ford-Fulkerson

b) Suppose (A', B') has $c(A', B') < c(A^*, B^*)$

Then $v(f) \le c(A', B')$ flow can't exceed capacity

But $v(f) = c(A^*, B^*)$
$> c(A', B')$ $\Rightarrow\Leftarrow$

$\therefore$ $c(A', B') \ge c(A^*, B^*)$

Ford-Fulkerson

MAX-FLOW-FF(G, s, t)

$f(u,v) \leftarrow 0$ for all edges (u,v)
$G_r \leftarrow G$ with $c_r(u,v) = c(u,v)$ and $c_r(v,u) = 0$ for all $(u,v) \in E$

while there is a path P $s \leadsto t$ in G_r s.t. all edges $(x,y) \in P$ have $c_r(x,y) > 0$
    $P \leftarrow$ one of those paths
    $b \leftarrow$ bottleneck(P)
    for $(x,y) \in P$
        if (x,y) is a forward edge
            $f(x,y) \leftarrow f(x,y) + b$
        else
            $f(y,x) \leftarrow f(y,x) - b$
        $c_r(x,y) \leftarrow c_r(x,y) - b$
        $c_r(y,x) \leftarrow c_r(y,x) + b$
return f

augment(G_r, f)

Polynomial Time Ford-Fulkerson

MAX-FLOW(G)

$f(u,v) \leftarrow 0$ for all $(u,v) \in E$
$G_r \leftarrow G$

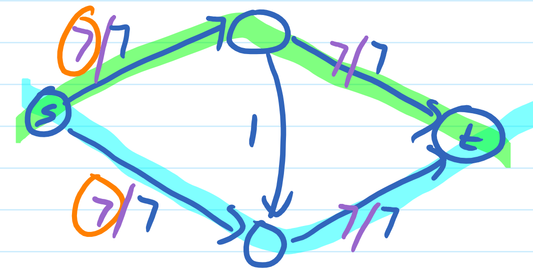
$O(C)$ iterations while there is a path P $s \rightsquigarrow t$ in G_r
$O(m)$ augment(G_r, f) $\leftarrow$ adds ≥ 1 unit of flow

$O(C \cdot m)$ total

$$C = \sum_{(s,v) \in E} c(s,v)$$

Polynomial Time Ford-Fulkerson

$v(f) = 7 + 7 = 14$



<u>MAX-FLOW(G)</u>

$f(u,v) \leftarrow 0$ for all $(u,v) \in E$
$G_r \leftarrow G$

$C = \sum_{(s,v) \in E} c(s,v)$

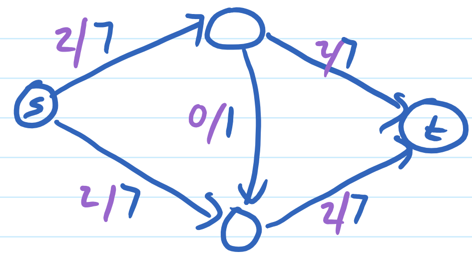
$O(C)$ iterations while there is a path P $s \rightsquigarrow t$ in G_r
$O(m)$ augment(G_r, f) $\leftarrow$ adds ≥ 1 unit of flow

$O(C \cdot m)$ total

Polynomial Time Ford-Fulkerson

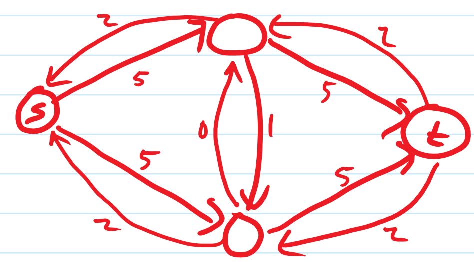
MAX-FLOW(G)

$f(u,v) \leftarrow 0$ for all $(u,v) \in E$
$G_r \leftarrow G$



$C = \sum_{(s,v) \in E} c(s,v)$

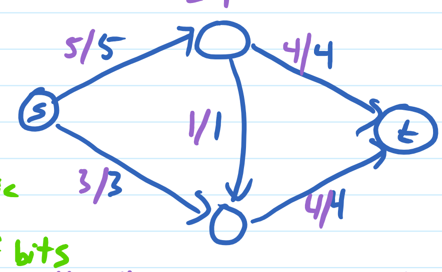
$O(C)$ iterations while there is a path P $s \rightsquigarrow t$ in G_r
$O(m)$ augment(G_r, f) $\leftarrow$ adds ≥ 1 unit of flow



$O(C \cdot m)$ total

Polynomial Time Ford-Fulkerson

Δ = lower bound on flow to add
$= 1$



MAX-FLOW (G)

$f(u,v) \leftarrow 0$ for all $(u,v) \in E$
$G_r \leftarrow G$

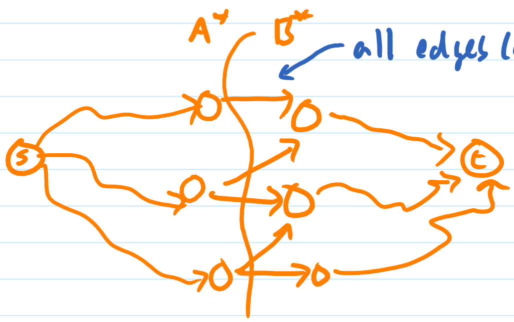
$\Delta = 2^{\lfloor \log_2 \max_{(s,v) \in E} c(s,v) \rfloor}$

# of iterations logarithmic in largest capacity, so linear (polynomial) in # of bits

while $\Delta \geq 1$
  while there is a path P $s \leadsto t$ in G_r with all edges having residual capacity $\geq \Delta$
   $O(m)$ augment(G_r, f, Δ) ← adds at least Δ to value of f
  $\Delta \leftarrow \frac{1}{2}\Delta$

INVARIANT: there is an s-t cut (A^*, B^*) s.t. $c(A^*, B^*) \leq v(f) + 2m\Delta$

$A^* = \{ v \mid \exists \text{ path } P \ s \leadsto v \text{ in } G_r \text{ s.t. } \forall (x,y) \in P,\ c_r(x,y) \geq \Delta \}$

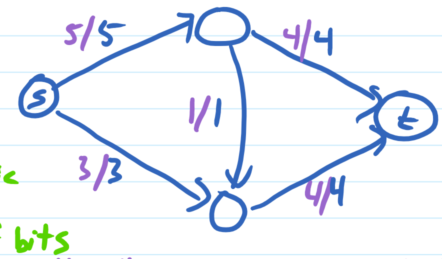


all edges (u,v) across have $c_r(u,v) < \Delta_{old} = 2\Delta_{new}$

$$v(f_{new}) = f_{new}^{out}(A^*) - f_{new}^{in}(A^*)$$
$$= \sum_{\substack{(u,v) \in E \\ u \in A^*, v \in B^*}} f_{new}(u,v) - \sum_{\substack{(u,v) \in E \\ u \in B^*, v \in A^*}} f_{new}(u,v)$$
$$> \sum_{\substack{(u,v) \in E \\ u \in A^*, v \in B^*}} \left(c(u,v) - 2\Delta_{new}\right) - \sum_{\substack{(u,v) \in E \\ u \in B^*, v \in A^*}} 2\Delta_{new}$$
$$= c(A^*, B^*) - 2m\Delta_{new}$$

Polynomial Time Ford-Fulkerson

Δ = lower bound on flow to add
= 1



MAX-FLOW(G)

$f(u,v) \leftarrow 0$ for all $(u,v) \in E$
$G_r \leftarrow G$

$\Delta = 2^{\lfloor \log_2 \max_{(s,v)\in E} c(s,v) \rfloor}$

while $\Delta \geq 1$ ← # of iterations logarithmic in largest capacity, so linear (polynomial) in # of bits

O(m) iter while there is a path P $s \leadsto t$ in G_r with all edges having residual capacity $\geq \Delta$

O(m) augment(G_r, f, Δ) ← adds at least Δ to value of f

$\Delta \leftarrow \frac{1}{2}\Delta$

INVARIANT: there is an s-t cut (A^*, B^*) s.t. $c(A^*, B^*) \leq v(f) + 2m\Delta$

O(m^2 · #bits in capacities) polynomial

$A^* = \{ v \mid \exists$ path P $s \leadsto v$ in G_r s.t. $\forall (x,y) \in P,\ c_r(x,y) \geq \Delta \}$

$$v(f_{new}) \leq c(A^*, B^*) \leq v(f_{old}) + 2m\Delta_{old}$$

$$v(f_{new}) - v(f_{old}) \leq 2m\Delta_{old}$$

each iteration of inner loop adds Δ_{old} flow; can't add more than $2m\Delta_{old}$
∴ at most 2m iterations of inner loop

Polynomial Time Ford-Fulkerson

<u>MAX-FLOW(G)</u>

$f(u,v) \leftarrow 0$ for all $(u,v) \in E$
$G_r \leftarrow G$

while there is a path P $s \rightsquigarrow t$ in G_r
  augment(P, f)
  update(G_r, P)

Polynomial Time Ford-Fulkerson

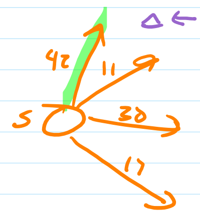
MAX-FLOW(G)

$f(u,v) \leftarrow 0$ for all $(u,v) \in E$

$G_r \leftarrow G$

$\Delta \leftarrow 2^{\lfloor \log_2 \max_{(s,v)\in E} c(s,v) \rfloor}$

$\Delta \leftarrow 32 \quad (32 \le 42)$



$O(\log_2 C)$ while $\Delta \ge 1$

iterations??? → while there is a path P $s \leadsto t$ in $G_{r,\Delta}$

$O(m)$ $\quad O(n)$ [augment(P, f) ; update(G_r, P)]

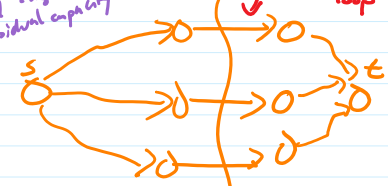
$\Delta \leftarrow \Delta / 2$

↑ residual graph w/ only edges w/ residual capacity $\ge \Delta$

total $\underline{O(\log_2 C \cdot m \cdot n)}$

polynomial in size of graph + # bits in capacities

residual capacity $< \Delta$ at end of inner loop



capacity of cut $< m \cdot \Delta$

INV (outermost loop): there is a cut (A, B) s.t. $c(A,B) \le v(f) + 2m \cdot \Delta$

Basis:

Maintenance: Let $A^* = \{v \mid s \leadsto v \text{ in } G_{r,\Delta}\}$, $B^* = V - A^*$

$v(f) = f^{out}(A^*) - f^{in}(A^*)$ THM of Mar 27

$= \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} f(u,v) - \sum_{\substack{(u,v)\in E \\ u \in B^* \\ v \in A^*}} f(u,v)$ def f^{out}, f^{in}

for edges $A^* \to B^*$

$c_r(u,v) = c(u,v) - f(u,v) < \Delta$

(if $c_r(u,v) \ge \Delta$ then $(u,v) \in G_{r,\Delta}$ so $s \leadsto u \to v$ $\Rightarrow\Leftarrow$)

$\ge \sum_{\substack{(u,v)\in E \\ u \in A^* \\ v \in B^*}} (c(u,v) - \Delta_{old}) - \sum_{\substack{(u,v)\in E \\ u \in B^* \\ v \in A^*}} \Delta_{old}$

for edges $\overset{u}{B^*} \to \overset{v}{A^*}$

$f(u,v) < \Delta$

(if $f(u,v) \ge \Delta$ then $c_r(v,u) \ge \Delta$ so $(v,u) \in G_{r,\Delta}$ and $s \leadsto v \to u$ $\Rightarrow\Leftarrow$)

$\ge c(A^*, B^*) - m\,\Delta_{old}$

$= c(A^*, B^*) - 2m\,\Delta_{new}$

any flow $\le$ capacity of any cut

INV

$v(f_{new}) \le v(f^*) \le c(A^*, B^*) \le v(f_{old}) + 2m\Delta$

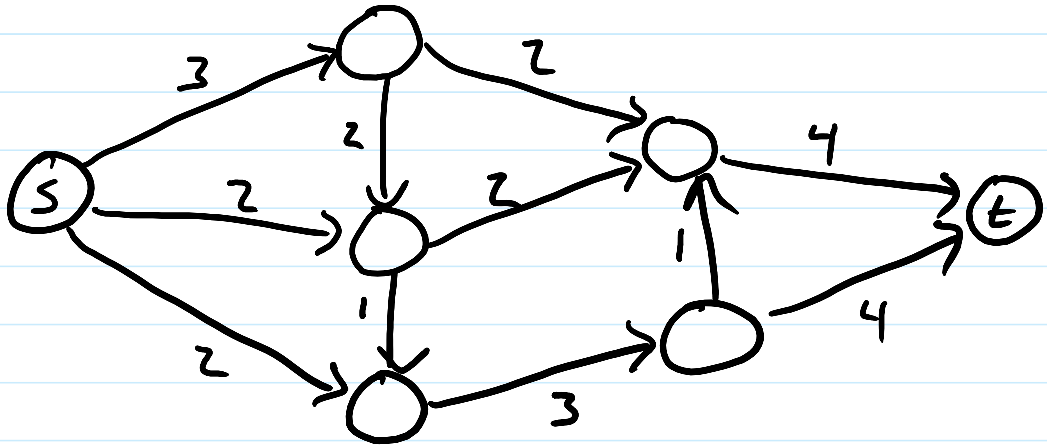
$v(f_{new}) - v(f_{old}) \le \underline{2m\Delta}$ $\quad m=10, \Delta=16 \quad 320$

we add at most this during 1 iteration of outer loop

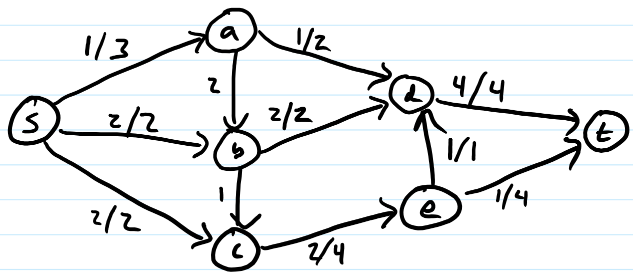
each iteration of inner adds $\ge \Delta$ flow

so $\le \underline{2m}$ iterations of inner

Quiz



Quiz



is there an augmenting path

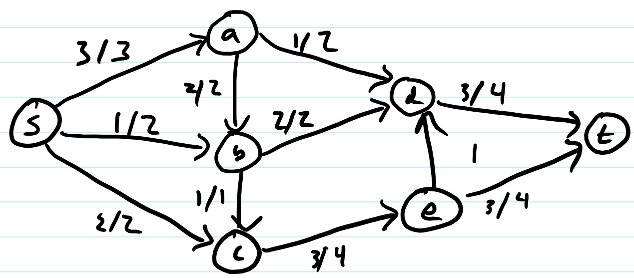
which is bottleneck

how much flow can we add

what is value of resulting flow

residual capacity on (s,a) (a,s)

is it maximum?



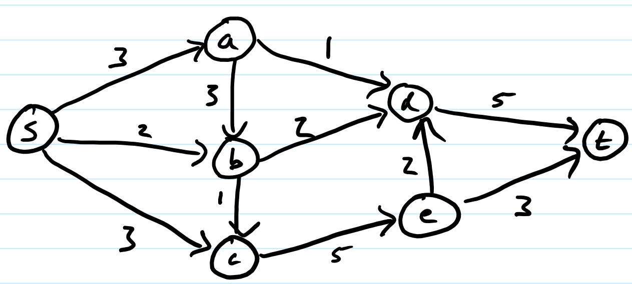
is there an augmenting path?

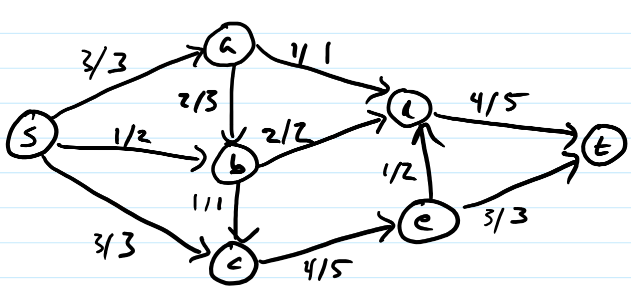
how much flow can we add?

is it maximum?

what is capacity of min-cap cut

what is the min-capacity cut?

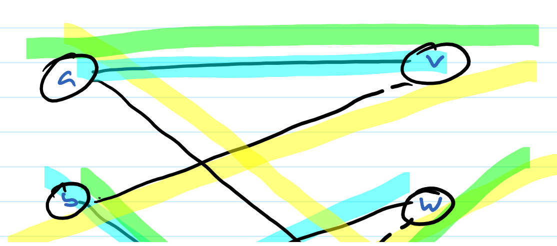




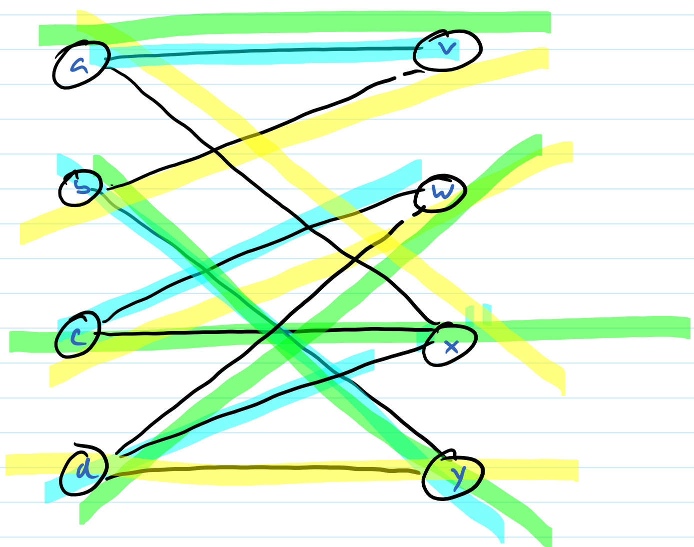
what is value of max flow

what is capacity of min-cut

what is min cut



green	blue	yellow
a – v	v	x
b – y	y	v
c – x	w	w
d – w	x	y



a – v	v	x
b – y	y	v
c – x	w	w
d – w	x	y