**NP-complete Problems**

Hamiltonian Cycle is NP-complete

TSP $\in$ NP and HC $\le_p$ TSP, so TSP is NP-complete
HP $\in$ NP and HC $\le_p$ HP, so HP is NP-complete

Vertex Cover is NP-complete

Independent Set $\in$ NP and VC $\le_p$ IS, so IS is NP-complete

3-SAT is NP-complete

Vertex Cover $\in$ NP

3-SAT $\le_p$ VC

$\therefore$ VC is NP-complete

conjunctive normal form — assignment of T/F to variables to make $\varphi$ T

3-SAT : given 3-CNF $\varphi$, determine if $\varphi$ has satisfying assignment

(3-CNF: conjunction of clauses, each a disjunct of up to 3 variables or negations of variables)

$(T \vee F \vee T) \wedge (F \vee T \vee F) \wedge (T \vee T \vee T) \wedge (F \vee T \vee T) = T$

satisfiable: $(x \vee y \vee z) \wedge (\sim x \vee \sim y \vee \sim z) \wedge (x \vee \sim y \vee z) \wedge (\sim x \vee \sim y \vee z)$

$x = T \quad y = F \quad z = T$

not satisfiable : $(x \vee y) \wedge (\sim x \vee y) \wedge (x \vee \sim y) \wedge (\sim x \vee \sim y)$

(up to 3 terms per clause)

# 3-SAT to VC

3-SAT $\le_p$ VC :

3-SAT($\varphi$)

G ← ???
k ← ???
result ← VC(G, k)
return result

construct G, pick k so that

$\varphi$ is satisfiable
$\Updownarrow$
G has vertex cover of size $\le k$

true terms = ones <u>not</u> in vertex cover of G

$$(x_1 \vee x_2 \vee \sim x_3) \wedge (\sim x_1 \vee x_3 \vee x_4) \wedge (\sim x_2 \vee \sim x_3 \vee \sim x_4)$$

# 3-SAT to VC

3-SAT $\le_p$ VC:

**3-SAT($\varphi$)**
G ← ???
k ← ???
result ← VC(G, k)
return result

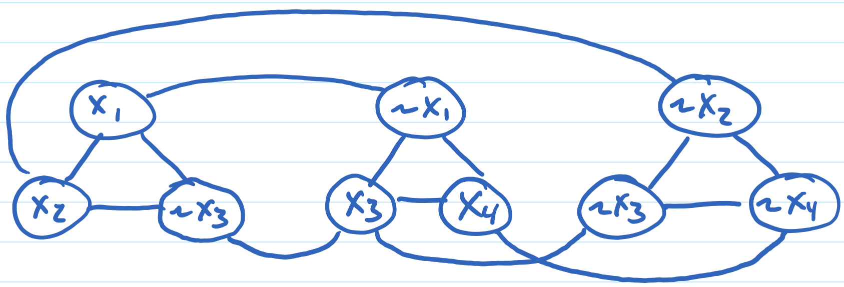
construct G, pick k so that

$\varphi$ is satisfiable
$\Updownarrow$
G has vertex cover of size $\le k$

true terms = ones <u>not</u> in vertex cover of G

$(x_1 \vee x_2 \vee \sim x_3) \wedge (\sim x_1 \vee x_3 \vee x_4) \wedge (\sim x_2 \vee \sim x_3 \vee \sim x_4)$

create 3 vertices/clause – 1 per term
let $k = 2m$
add edges between verts in same clause
add edges between $x_i$ and $\sim x_i$



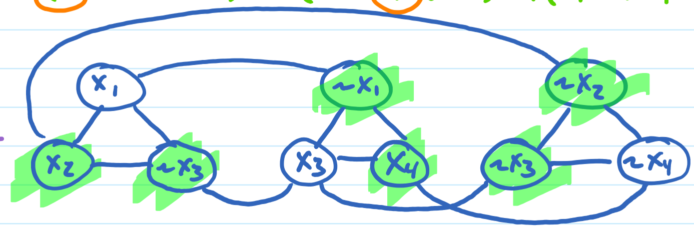
# 3-SAT to VC

$x_1 = T \quad x_2 = T \quad x_3 = T \quad x_4 = F$

$(x_1 \vee x_2 \vee \sim x_3) \wedge (\sim x_1 \vee x_3 \vee x_4) \wedge (\sim x_2 \vee \sim x_3 \vee \sim x_4)$

$(T \vee T \vee F) \wedge (F \vee T \vee F) \wedge (F \vee F \vee T) = T$

create 3 vertices/clause – 1 per term
let $k = 2m$
add edges between verts in same clause
add edges between $x_i$ and $\sim x_i$




$\varphi$ has a satisfying assignment $A \rightarrow G$ has a vertex cover $C$ with $|C| \leq k = 2m$

A makes at least one term per clause T

Construct S by picking <u>one</u> vertex per clause whose term is made T by A

Let $C = V - S$. Then $|C| = 2m$ (since $|S| = m$ and $|V| = 3m$)

C covers all edges in triangles (C contains 2 vertices per △)
C covers all edges between triangles (if not covered, then some $x_i$ and $\sim x_i$ are both in S so both T)

C is a vertex cover of size $k = 2m$

**3-SAT to VC**

$x_1 = F \quad x_2 = T \quad x_3 = F \quad x_4 = T$

$(x_1 \vee x_2 \vee \sim x_3) \wedge (\sim x_1 \vee x_3 \vee x_4) \wedge (\sim x_2 \vee \sim x_3 \vee \sim x_4)$

$(F \vee T \vee T) \wedge (T \vee F \vee T) \wedge (F \vee T \vee F) = T$

✓ $3m$ vertices

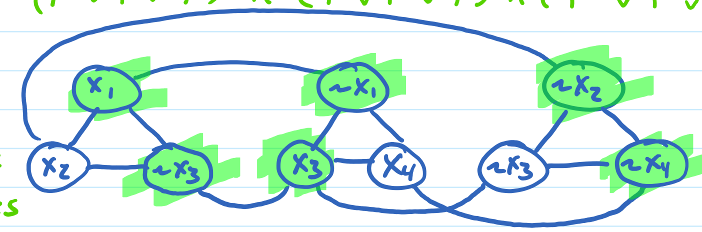
create 3 vertices/clause – 1 per term

let $k = 2m$

add edges between verts in same clause

add edges between $x_i$ and $\sim x_i$ $\leq (3m)^2$ edges

polynomial time




$G$ has a vertex cover $C$ with $|C| \leq k = 2m \rightarrow \varphi$ has a satisfying assignment $A$

$C$ contains $\geq 2$ vertices per triangle (otherwise can't cover all edges in $\Delta$)

$C$ contains $\leq 2$ vertices per triangle (otherwise has size $> 2m$)

$C$ contains 2 vertices per triangle

Create $A$ so that $A(x_i) = T$ if any vertex labelled $x_i$ <u>not</u> in $C$ (and others arbitrarily)

$A(x_i) = F$ if any vertex labelled $\sim x_i$ <u>not</u> in $C$

$A$ is an assignment ($A(x_i)$ is both $T$ and $F$ means $C$ fails to cover some $(x_i, \sim x_i)$)

$A$ is a satisfying assignment (1 vertex per triangle/clause not in $C$ so is made $T$)