Dirk Bergemann — Yale University
Joan Feigenbaum

Economics and Computation

Computer Science 455/555 and Economics 425/563

First Part of Problem Set 1:
Game Theory and Mechanism Design
9/12/08

The entire problem set, First and Second Part, are due on Thursday, 10/02/08.

1. **Practice with Normal Form Games.** Consider the following normal form (or strategic form) game:

	L	R
U	$1, -1$	$3, 0$
D	$4, 2$	$0, -1$

   1. Compute each player's reaction correspondence as a function of his opponent's randomizing probability.
   2. For which probabilities is player i indifferent between his two strategies regardless of the play of his opponent?
   3. Derive the best-response correspondences graphically by plotting player $i's$ payoff to his two pure strategies as a function of his opponent's mixed strategy.
   4. Plot the two reaction correspondences in the (x, y) space. What are the Nash equilibria?

2. **The Tragedy of the Commons (after Hume).** Suppose there are I farmers, each of whom has the right to graze cows on the village common. The amount of milk a cow produces depends on the total number of cows, N, grazing on the green. The revenue produced by n_i cows is $n_i v(N)$ where $v(0) > 0$, $v'(N) \leq 0$ and $v''(N) \leq 0$ for all $N \in \mathbb{R}_+$, also $v'(N) < 0$ for $N < \bar{N}$ and $v(N) = 0$ for all $N \geq \bar{N}$ for some large $\bar{N} \gg 0$. Each cow costs c, and cows are perfectly divisible. Suppose $v(0) > c$. Farmers simultaneously decide how many cows to purchase; all purchased cows will graze on the common.

   1. Write this as a game in strategic form.

2. Find the condition for the socially optimal number of cows i.e. N which maximizes $Nv(N) - CN$.
3. Find the Nash equilibrium in which each of the farmers makes an individual decision and compare it against the social optimum.

3. **Equivalence of Equilibrium Conditions.** In class, we claimed that the following two inequalities describing a best response of agent i and therefore the equilibrium conditions are equivalent:

$$\sum_{t \in T} p(t)\, u_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), t\right) \geq \sum_{t \in T} p(t)\, u_i\left(\left(s_i(t_i), s_{-i}^*(t_{-i})\right), t\right) \qquad (0.1)$$
$$\text{for all } s_i \in S_i \text{ and } i = 1, .., I$$

and after writing

$$p(t_i) = \sum_{t'_{-i}} p\left(t_i, t'_{-i}\right)$$

and

$$p(t_{-i}|t_i) \equiv \frac{p(t_i, t_{-i})}{p(t_i)}$$

the condition:

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i)\, u_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), t\right) \geq \sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i)\, u_i\left(\left(a_i, s_{-i}^*(t_{-i})\right), t\right)$$
$$\text{for all } t_i \in T_i,\ a_i \in A_i \text{ and } i = 1, ..., I \qquad (0.2)$$

1. Now argue formally that they are equivalent. (Hint: An argument by contradiction should suffice.)
2. By comparing the conditions (0.1) and (0.2), explain verbally the difference between a strategy s_i and an action a_i.

4. **Bilateral Trade.** Consider a bilateral trade setting in which both the seller's (agent 1) and the buyer's (agent 2) types are drawn independently from the uniform distribution on [0,1]. Suppose that we try to implement the social choice function $f(\cdot) = (y_1(\cdot), y_2(\cdot), t_1(\cdot), t_2(\cdot))$ such that

$$\begin{aligned}
y_1(\theta_1, \theta_2) &= 1 \text{ if } \theta_1 \geq \theta_2; = 0 \text{ if } \theta_1 < \theta_2. \\
y_2(\theta_1, \theta_2) &= 1 \text{ if } \theta_2 > \theta_1; = 0 \text{ if } \theta_2 \leq \theta_1. \\
t_1(\theta_1, \theta_2) &= \tfrac{1}{2}(\theta_1 + \theta_2) y_2(\theta_1, \theta_2). \\
t_2(\theta_1, \theta_2) &= -\tfrac{1}{2}(\theta_1 + \theta_2) y_2(\theta_1, \theta_2).
\end{aligned}$$

1. Illustrate the allocation rule y in a two dimensional graph (θ_1, θ_2).
2. Suppose that the seller truthfully reveals his type for all $\theta_1 \in [0,1]$. Will the buyer find it worthwhile to reveal his type? Interpret.
3. Consider the *double auction* mechanism in which the seller (agent 1) and the buyer (agent 2) each submit a sealed bid, $b_i \geq 0$. If $b_1 \geq b_2$, the seller keeps the good and no monetary transfer is made; while if $b_2 > b_1$, the buyer gets the good and pays the seller the amount $\frac{1}{2}(b_1 + b_2)$. (The interpretation is that the seller's bid is his minimum acceptable price, while the buyer's is his maximum acceptable price; if trade occurs, the price splits the difference between these amounts.) Solve for a Bayesian Nash equilibrium of this game in which each agent i's strategy takes the form $b_i(\theta_i) = \alpha_i + \beta_i \theta_i$. What social choice function does this equilibrium of this mechanism implement? Illustrate the equilibrium outcome in the two-dimensional graph. Is it ex post efficient?
4. Show that the social choice function derived in **(7.3)** is incentive compatible; that is, that it can be truthfully implemented in Bayesian Nash equilibrium in the direct mechanism.

Readings. (Nisan, Roughgarden, Tardos & Vazirani 2008), Chapters 1 and 9.
History of Game Theory. Perhaps the two most influential persons in the development of game theory are John von Neumann and John Nash. The biographies by (Poundstone 1992) and (Nasar 1998), respectively, make for an excellent bedtime reading into their fascinating personal and intellectual lives. A more formal evaluation of the impact of Nash's work is given by (Myerson 1999) and on the first three Nobel prize winners in game theory by (Gul 1997).

References

Gul, F. 1997. "A Nobel Prize for Game Theorists: The Contributions of Harsanyi, Nash and Selten." *Journal of Economic Perspectives* 11:159–174.

Myerson, R.B. 1999. "Nash Equilibrium and the History of Economic Theory." *Journal of Economic Literature* 37:1067–1082.

Nasar, S. 1998. *A Beautiful Mind: A Biography of John Forbes Nash, Jr.* New York: Simon and Schuster.

Nisan, N., T. Roughgarden, E. Tardos & V. Vazirani. 2008. *Alogrithmic Game Theory.* Cambridge: Cambridge University Press.

Poundstone, W. 1992. *Prisoner's Dilemna.* New York: Doubleday.