Part 2 of Homework 1 in ECON 425/563 // CPSC 455/555

See http://zoo.cs.yale.edu/classes/cs455/2008/HW1a.pdf for part 1 of this homework assignment. Both parts are due on October 2, 2008. Each of the four problems in part 1 and the four problems below is worth 12 points, for a total of 96. (Everyone gets 4 points for free. :=))

5. **Coping with NP-hardness**. Recall that the following languages are NP-complete.
CLIQUE = {(G, k), where G = (V, E) is an undirected graph, k is a positive integer, and there is a subset V' of V such that $|V'| \geq k$ and (u, v) ε E for all pairs u, v of distinct nodes in V'}.
HC = {G, where G = (V, E) is an undirected graph, |V| = n, and there is an ordering $<v_1, \ldots, v_n>$ of the nodes in V such that (v_n, v_1) ε E and (v_i, v_{i+1}) ε E, for $1 \leq i \leq n-1$}.
COLORABLE = {(G, k), where G = (V, E) is an undirected graph, k is a positive integer, and there is a function f: V → {1, 2, …, k} such that f(u) ≠ f(v) if (u, v) ε E}.

Prove that the following special cases of these language are in P.

(a) (2 points) k_0-CLIQUE = {G, where G = (V, E) is an undirected graph, and there is a subset V' of V such that $|V'| \geq k_0$ and (u, v) ε E for all pairs u, v of distinct nodes in V'}. (Here, k_0 is a fixed, positive integer, i.e., one that does not depend on the size of the input graph.)
(b) (4 points) DEGREE-2-HC = {G, where G = (V, E) is an undirected graph; |V| = n; for each u ε V, there are at most two other nodes v and w such that (u, v) ε E and (u, w) ε E; and there is an ordering $<v_1, \ldots, v_n>$ of the nodes in V such that (v_n, v_1) ε E and (v_i, v_{i+1}) ε E, for $1 \leq i \leq n-1$}
(c) (6 points) 2-COLORABLE = {G, where G = (V, E) is an undirected graph, and there is a function f: V → {1, 2} such that f(u) ≠ f(v) if (u, v) ε E}

6. **Basic complexity classes**. Prove that $P \subseteq NP \subseteq PSPACE$.

7. **A game on directed graphs**. The language GEOGRAPHY is defined as follows. An instance consists of a directed graph G = (V, A) and a designated start node s ε V. Player I moves first by choosing node s; then player II moves by choosing a node s' ≠ s such that (s, s') ε A. More generally, after m moves have been made, exactly m nodes have been chosen, and one of the two players has chosen node u in the m^{th} move; the $(m+1)^{st}$ move is then made by the other player, who must choose a node v such that (u, v) ε A, and v has not already been chosen in one of the first m moves. When a player is unable to move (because no such node v exists), he loses. The instance (G, s) is a yes-instance of GEOGRAPHY if and only if player I has a winning strategy.

(a) (1 points) Construct a yes-instance of GEOGRAPHY.
(b) (1 point) Construct a no-instance of GEOGRAPHY.
(c) (10 points) Prove that GEOGRAPHY is in PSPACE.

8. **Computing equilibria in games**: Give an algorithm that takes as input a two-player game in normal form and produces as output a Nash Equilibrium of the game. (You should use the definitions of "game in normal form" and "Nash Equilibrium" that were given in Lecture I on September 9, 2008.) You need not give a polynomial-time algorithm, but you must explain what the size n of a problem instance is and give upper bounds on the time complexity T(n) and the space complexity S(n) of your algorithm. It may be useful first to describe an algorithm that finds a pure-strategy Nash equilibrium if one exists and then modify it to accommodate the possibility that a mixed-strategy Nash equilibrium is needed.