Maximum Matching in graphs

We describe Edmond's Matching Algorithm to find the maximum (cardinality) matching in a general undirected graph $G(V, E)$. There are two important basic Lemmas we need first.

Definition 1 (Alternating Path) *An alternating path with respect to a matching M in G is a path consisting alternately of edges in M and edges not in M.*

An *augmenting path* with respect to M is an alternating path with both end points unmatched in M. (Why the name ?)

Lemma 1 *(Berge) A matching M is maximum iff it admits no augmenting path.*

One direction is clear. Suppose M is not maximum. Let M^* be a maximum matching. Look at the symmetric difference between M and M^*. It consists of disjoint cycles and paths (why?). One of the paths must be an augmenting path.

Definition 2 *A blossom with respect to a matching M is an odd cycle on say $2k + 1$ vertices containing k edges of M. The unique vertex of the cycle with no edge of M in the cycle incident to it is called the base of the blossom.*

Lemma 2 (Blossoms) *Let M be a matching and C be a blossom with respect to M. Suppose there is an alternating path Q of even length from a vertex v not covered by M to the base of C, where Q and C have no edges in common. Let G', M' result from G, M respectively by shrinking $V(C)$ to a single vertex. Then M admits an augmenting path in G iff M' admits an augmenting path in G'.*

The algorithm will try to find an augmenting path essentially by building a special type of breadth-first- search forest. It will either find one, or find a blossom as in the Lemma above or conclude that the current matching is maximum.

The forest starts out with the set S of vertices which are unmatched in M and no edges. [These are all potential ends of augmenting paths.] We call them level 1 vertices (of the bfs forest) or the "roots".

At a general stage, we consider a vertex x at an odd level (say $2k + 1$) and examine all the edges from x. Suppose y is adjacent to x. There are three interesting cases :

(i) y is not in the forest. Then we grow the forest by adding y at level $2k + 2$ and the unique edge (y, z) of M incident to y (why is there one ?) as well as z at level $2k + 3$. y, z are in the tree rooted at the same node of S as x is.

(ii) y is in the forest already at an odd level in a different connected component (than x) . Then, the path from the root of x to x, then (x, y) plus the path from y to its root forms an augmenting path. Stop.

(iii) y is in the forest already at an odd level in the same connected component as x. We have found a blossom satisfying the Lemma above. Shrink the blossom. [Problem reduced to a smaller one.]

(iv) The only other possibility is that y is already in the forest at an even level. In this case, we do not add the edge (x, y) to the forest.

In case (iii) ever occurs, it is clear that we have a blossom satisfying the Lemma. So now assume that (iii) never occurs. Also, if (ii) ever occurs, we are clearly done. So, assume not. Then, when we stop, we must have that for all odd level vertices, their neighbours are all at even levels. Then we claim that M is a maximum matching. To see this, note that if we remove all the even level vertices from the graph, there is no edge left. (Why ?) So, in any matching, any matched odd level vertex must be matched to an even level vertex, so the number of edges in the matching is at most the number of even level vertices. $|M|$ eqauls the number of even level vertices, so it must be maximum.

Running time It is easy to see that if $T(n)$ is the maximum time taken on any $n-$ node graph, we have

$$T(n) \leq T(n-2) + p(n),$$

where $p(\cdot)$ is a polynomial. (Why ?) Thus $T(n)$ is bounded by a polynomial. There have been many improvements.