CPSC 467: Cryptography and Computer Security

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Oblivious Transfer

Locked-box protocol revisited

In the locked box coin-flipping protocol, Alice has two messages m_0 and m_1.

Bob gets one of them.

Alice doesn't know which (until Bob tells her).

Bob can't cheat to get both messages.

Alice can't cheat to learn which message Bob got.

The *oblivious transfer problem* abstracts these properties from particular applications such as coin flipping and card dealing,

Oblivious Transfer of One Secret

Oblivious transfer of one secret

Alice has a secret s.

An oblivious transfer protocol has two equally-likely outcomes:

1. Bob learns s.
2. Bob learns nothing.

Afterwards, Alice doesn't know whether or not Bob learned s.

A cheating Bob can do nothing to increase his chance of getting s.

A cheating Alice can do nothing to learn whether or not Bob got her secret.

Rabin proposed an oblivious transfer protocol based on quadratic residuosity in the early 1980's.

Rabin's OT protocol

	Alice		Bob
1.	Secret s. $n = pq$, $p \neq q$ prime. RSA public key (e, n). Compute $c = E_{(e,n)}(s)$.	$\xrightarrow{(e,n,c)}$	
2.		$\xleftarrow{a}$	Choose random $x \in \mathbf{Z}_n^*$. Compute $a = x^2 \bmod n$.
3.	Check $a \in \mathrm{QR}_n$. Random $y \in \sqrt{a} \pmod n$.	$\xrightarrow{y}$	
4.			Check $y^2 \equiv a \pmod n$. If $y \not\equiv \pm x \pmod n$, use x, y to factor n and decrypt c to obtain s.

Analysis

Alice can can carry out step 3 since she knows the factorization of n and can find all four square roots of a.

However, Alice has no idea which x Bob used to generate a.

Hence, with probability $1/2$, $y \equiv \pm x \pmod{n}$ and with probability $1/2$, $y \not\equiv \pm x \pmod{n}$.

If $y \not\equiv \pm x \pmod{n}$, then the two factors of n are $\gcd(x - y, n)$ and $n/\gcd(x - y, n)$, so Bob factors n and decrypts c in step 4.

However, if $y \equiv \pm x \pmod{n}$, Bob learns nothing, and Alice's secret is as secure as RSA itself.

A potential problem

There is a potential problem with this protocol.

A cheating Bob in step 2 might send a number a which he generated by some means other than squaring a random x.

In this case, he always learns something new no matter which square root Alice sends him in step 3.

Perhaps that information, together with what he already learned in the course of generating a, is enough for him to factor n.

Is this a real problem?

We don't know of any method by which Bob can find a quadratic residue $a \pmod{n}$ without also knowing one of a's square roots.

We certainly don't know of any method that would produce a quadratic residue a together with some other information Ξ that, combined with a square root y, would allow Bob to factor n.

But we also cannot prove that no such method exists.

A modified protocol

We fix this problem by having Bob prove that he knows a square root of the number a that he sends Alice in step 2.

He does this using a zero knowledge proof of knowledge of a square root of a.

This is essentially what the simplified Feige-Fiat-Shamir protocol of Lecture 16 does, but with the roles of Alice and Bob reversed.

- Bob claims to know a square root x of the public number a.
- He wants to prove to Alice that he knows x, but he does not want Alice to get any information about x.
- If Alice were to learn x, then she could choose $y = x$ and eliminate Bob's chance of learning s while still appearing to play honestly.

Oblivious Transfer of One Secret out of Two

One-of-two oblivious transfer

In *one-of-two oblivious transfer*, Alice has two secrets, s_0 and s_1.

Bob always gets exactly one of the secrets, each with probability 1/2.

Alice does not know which one Bob gets.

The locked box protocol is one way to implement one-of-two oblivious transfer.

Another is based on a public key cryptosystem (such as RSA) and a symmetric cryptosystem (such as AES).

This protocol given next does not rely on the cryptosystems being commutative.

Intuitive idea

In this protocol, Alice chooses two PKS key pairs and sends the public keys to Bob.

Bob chooses a random key k for a symmetric cryptosystem, encrypts it with one of Alice's two keys chosen at random, and sends the ciphertext to Alice.

Alice decrypts Bob's ciphertext using both of her decryption keys and obtains two numbers $\{k_0, k_1\}$. One of them is Bob's k; the other is garbage. She encrypts one secret using k_0 and one using k_1 and sends to Bob.

Bob decrypts the one that was encrypted with k.

A one-of-two OT protocol

	Alice		Bob
1.	Secrets s_0 and s_1. Choose two PKS key pairs (e_0, d_0) and (e_1, d_1).	$\xrightarrow{e_0, e_1}$	
2.		$\xleftarrow{c}$	Choose random key k for symmetric cryptosystem $(\hat{E}, \hat{D})$. Choose random $b \in \{0,1\}$. Compute $c = E_{e_b}(k)$.
3.	Let $k_i = D_{d_i}(c)$, $i \in \{0,1\}$. Choose $b' \in \{0,1\}$. Let $c_i = \hat{E}_{k_i}(s_{i \oplus b'})$, $i \in \{0,1\}$.	$\xrightarrow{c_0, c_1}$	
4.			Output $s = s_{b \oplus b'} = \hat{D}_k(c_b)$.

Analysis

In step 2, Bob encrypts a randomly chosen key k for the symmetric cryptosystem using one of the PKS encryption keys that Alice sent him in step 1.

He then selects one of the two encryption keys from Alice, uses it to encrypt k, and sends the encryption to Alice.

In step 3, Alice decrypts c using both decryption keys d_0 and d_1 to get k_0 and k_1.

One of the k_i is Bob's key k (k_b to be specific) and the other is garbage, but because k is random and she doesn't know b, she can't tell which is k.

Analysis (cont.)

She then encrypts one secret with k_0 and the other with k_1, using the random bit b' to ensure that each secret is equally likely to be encrypted by the key that Bob knows.

In step 4, Bob decrypts the ciphertext c_b using key his key $k = k_b$ to recover the secret $s = s_{b \oplus b'}$.

He can't decrypt the other ciphertext $c_{1 \oplus b}$ since he doesn't know the key $k_{1 \oplus b}$ used to produce it, nor does he know the decryption key $d_{1 \oplus b}$ that would allow him to find it from c.

A subtle security problem

Unfortunately, this protocol has a subtle security problem. We claimed that Alice doesn't know Bob's value b after step 2. But why should that be true?

c is the encryption of k using E_{e_0} or E_{e_1}. We would need to know that outputs of $E_{e_0}(k)$ for random k are indistinguishable from outputs of $E_{e_1}(k)$. We have no grounds for believing that.

For example, we could take our favorite PKS and construct a new one where $E'_e(k) = e \cdot E_e(k)$. This is just as secure as the original since e is already known to the adversary. However, the ciphertext here reveals the public key used for encryption.

Another 1-of-2 OT protocol using blinding [1]

	Alice		Bob
1.	Secrets s_0 and s_1. Choose RSA key (n, e, d). Let $y_i = E_e(s_i)$, $i \in \{0, 1\}$.	$\xrightarrow{n, e, y_0, y_1}$	
2.		$\xleftarrow{c}$	Choose random $b \in \{0, 1\}$. Choose random $r \in \mathbf{Z}_n^*$. Compute $c = y_b E_e(r) \bmod n$.
3.	Let $c' = D_d(c) \equiv s_b r \pmod{n}$.	$\xrightarrow{c'}$	
4.			Output $c' r^{-1} \bmod n = s_b$.

[1] This protocol is adapted from notes by David Wagner, U.C. Berkeley, CS276, lecture 29, May 2006.

Analysis

This protocol is much simpler.

- In step 1, Alice sends Bob encryptions of both secrets.
- In step 2, Bob chooses one of Alice's encryptions, blinds it, and returns the result to Alice.
- In step 3, Alice decrypts whatever Bob sends her, which allows Bob to unblind the decryption and recover the secret he chose in step 2.

Alice's other secret is safe assuming semi-honest parties (see lecture 20) as long as RSA is secure under a limited chosen ciphertext attack (since that is what Alice permits in step 3).

Bob's blinding prevents Alice from knowing which secret he learned.

Privacy-Preserving Multiparty Computation

Privacy

We have looked at many protocols whose goal is to keep Alice's information secret from an adversary, or sometimes even from Bob himself.

We now look at other protocols whose goal is to control the release of information about Alice's secret. Just enough information should be released to carry out the purpose of the protocol but no more.

This will become clearer with an example.

The Millionaire's Problem

The Millionaire's Problem

Alice and Bob want to know who is the richer without revealing how much they are actually worth.

Alice is worth I million dollars; Bob is worth J million dollars.

They want to determine whether or not $I \geq J$, but at the end of the protocol, neither should have learned any more about the other person's wealth than is implied by the truth value of the predicate $I \geq J$.

Privacy-preserving multiparty computation

The Millionaire's problem, introduced by Andy Yao in 1982, began the study of *privacy-preserving multiparty computation*.

Another example is vote-counting.

Each voter has an input $v_i \in \{0, 1\}$ indicating their no/yes vote on an issue.

The goal is to collectively compute $\sum v_i$ while maintaining the privacy of the individual v_i.

A solution to Yao's problem

For simplicity, assume that $I, J \in \{1, 2, \ldots, 10\}$.

Let N be a security parameter, and assume that Alice has public and private RSA keys (e, n) and (d, n), respectively, where $n = \bar{p}\bar{q}$, and $|\bar{p}| \approx |\bar{q}| \approx \frac{N}{2}$.

A protocol that intuitively works is shown on the next slide.[2]

[2] Adapted from web page "Solution to the Millionaire's Problem".

The protocol

Alice		Bob
1.		Choose x of length N. Let $C = E_{(e,n)}(x)$.
	$\xleftarrow{m}$	Let $m = (C - J + 1) \bmod n$.
2a. $Y_i = D_{(d,n)}(m + i - 1)$, $i \in [1, 10]$. [Note: $Y_J = x$.]		
2b. Choose prime p of length $N/2$ s.t. $\lvert Z_i - Z_j \rvert \geq 2$ for $i \neq j$, where $Z_i = (Y_i \bmod p)$, $i \in [1, 10]$.		
2c. Let $W_i = (Z_i + (i > I)) \bmod p$, $i \in [1, 10]$.	$\xrightarrow{p, W_1, \ldots, W_{10}}$	
3.	$\xleftarrow{\text{result}}$	$\text{result} = (W_J \equiv x \pmod p)$.

Verbal description

Alice decrypts $m, m+1, \ldots, m+9$ to get $Y_1, \ldots, Y_{10}$.

Y_J is Bob's secret, x, but Alice doesn't know which it is since all of the Y_i's "look" random.

She reduces them all mod random prime p to get $Z_1, \ldots, Z_{10}$. Note that $Z_J = x \bmod p$ and the other Z_i's look random.

Finally, she adds 1 $(\bmod p)$ to each of the numbers Z_i for which i is greater than her own wealth I. If she adds 1 to Z_J, this means that $J > I$; if not $J \leq I$.

Bob can tell which is the case from the numbers that Alice sends him in step 2c. Namely, if $W_J \equiv x \pmod p$, this means that 1 was not added, so $I \geq J$. Otherwise, $I < J$.

Detailed description

Alice		Bob
1.		Choose x of length N. Let $C = E_{(e,n)}(x)$.
	$\xleftarrow{m}$	Let $m = (C - J + 1) \bmod n$.
2a. $Y_i = D_{(d,n)}(m + i - 1)$, $i \in [1, 10]$. [Note: $Y_J = x$.]		

$C = (m + J - 1) \bmod n$ encrypts Bob's random secret x.

The numbers in $\mathcal{M} = \{m \bmod n, \ldots, (m + 9) \bmod n\}$ are "random-looking," and all are possible ciphertexts. Why?

Alice knows that $C \in \mathcal{M}$ but doesn't know which element it is.

After decryption, she knows that some $Y_i = x$ but not which one.

Detailed description (cont.)

Alice	Bob
2b. Choose prime p of length $N/2$ s.t. $\lvert Z_i - Z_j \rvert \geq 2$ for $i \neq j$, where $Z_i = (Y_i \bmod p)$, $i \in [1, 10]$.	

The numbers in $\mathcal{Y} = \{Y_1, \ldots, Y_{10}\}$ have no particular pattern. In all likelihood, no pair are at all close together.

Similarly, for most choices of p, no pair of Z_j's will be close.

Detailed description (cont.)

	Alice		Bob
2c.	Let $W_i = (Z_i + (i > I)) \bmod p$, $i \in [1, 10]$.	$\xrightarrow{p, W_1, \ldots, W_{10}}$	
3.		$\xleftarrow{\text{result}}$	$\text{result} = (W_J \equiv x \pmod p)$.

The W_i's are distinct and separated by at least 2, so j is the unique i such that $(W_i - x) \bmod p \in \{0, 1\}$. I don't know why this uniqueness condition is needed. Perhaps the intention is that Alice shuffle the W_i before sending.

Since $Y_J = x$, then $Z_J \equiv x \pmod p$.

Hence, if $W_J \equiv x \pmod p$, then $J \leq I$, otherwise $J > I$.

Privacy

Clearly, all that Alice learns from Bob is a set of random-looking numbers $m, \ldots, m+9$, one of which corresponds to Bob's wealth J, but she has no way of telling which, since any number in $\mathbf{Z}_n^*$ is the RSA encryption of some plaintext message.

Bob on the other hand receives p and $W_1, \ldots, W_{10}$ from Alice in step 2. However, he does not know any Z_i for $i \neq J$ since he cannot decrypt the corresponding numbers $m + i - 1$.

He also cannot recover Y_i from W_i because of the information loss implicit in the "mod p" operation. Thus, he also learns nothing about Alice's wealth I except for the value of the predicate $I \geq J$.

We remark that this protocol works only in the semi-honest model in which both Alice and Bob follow their protocol, but both will try to infer whatever they can about the others secrets after the fact.

A General Security Model

How can we define multiparty security?

How to define security in a multiparty protocol is far from obvious.

For example, in the millionaire's problem, there is no way to prevent either Alice or Bob from lying about their wealth, nor is it possible to prevent either of them from voluntarily giving up secrecy by broadcasting their wealth.

Thus, we can't hope to find a protocol that will prevent all kinds of cheating.

Ideal versus real protocol security model

What we do instead is to compare a given "real" protocol with a corresponding very simple "ideal" protocol involving a trusted third party.

The real protocol should simulate the ideal protocol, much the same as the simulator of a zero knowledge proof system simulates the real interaction between prover and verifier.

The real protocol is deemed to be secure if any bad things that can happen in the real protocol are also possible in the ideal protocol.

Example of an ideal protocol

The ideal protocol for the millionaire's problem has just two steps:

- Step 1: Alice and Bob send their secrets I and J, respectively, to the trusted party across a private, secure channel.
- Step 2: the trusted party computes the value of the predicate $I \geq J$ and sends the result back to both Alice and Bob.

The goal of the real protocol is that Alice and Bob don't learn any more than they could learn in the ideal protocol.

What does an ideal protocol compute?

What does an ideal multiparty protocol compute? Suppose there are m parties to the protocol, $P_1, \ldots, P_m$.

Each P_i has a private input x_i and receives a private output y_i.

We say that F is a (multiparty) *functionality* if F is a random process that maps m inputs to m outputs.

As a special case, we say that F *is deterministic* if the m outputs are uniquely determined by the m inputs.

The millionaire's problem can be expressed succinctly as the problem of securely computing the (deterministic) functionality

$$F(I, J) = ((I \geq J), (I \geq J))$$

in the semi-honest model.

Simple application of oblivious transfer

Consider the problem of privately evaluating a Boolean function $f(x, y)$, where x is private to Alice and y is private to Bob. This corresponds to privately computing the functionality

$$F(x, y) = (f(x, y), f(x, y)).$$

We use a slight variant of the one-out-of-two secrets oblivious transfer protocol presented last time:

In OT_1^2, the secrets are numbered s_0 and s_1. Bob requests and gets the secret of his choice, but Alice does not learn which secret he got.

This can be generalized to the case k secrets, where OT_1^k lets Bob choose one out of k.

The protocol

Here's the protocol.

1. Alice, with private input $x \in \{0, 1\}$, prepares a table T:

y	$f(x, y)$
0	$f(x, 0)$
1	$f(x, 1)$

   She doesn't know y, but she does know that the correct value $f(x, y)$ is in her table. It's either $f(x, 0)$ or $f(x, 1)$.
2. Bob, with private input y, obtains line y of the table using OT_1^2. Bob outputs $f(x, y)$ without learning x.
3. Bob sends $f(x, y)$ to Alice, who also outputs it.

Remarks

While this functionality seems almost too trivial to be interesting, it's really not.

For example, if $f(x, y) = x \wedge y$ and Alice knows $x = 0$, then the answer $f(x, y)$ does not tell her Bob's value y, so it's important that the protocol also not leak y in this case.

Similarly, when Bob requests the value corresponding to row 0, he gets no information about x when the result $f(x, 0) = 0$ comes back.

(In fact he knew that already before getting row 0 from Alice.)

Privacy-preserving Boolean Function Evaluation

Boolean functions computed by circuits

Let $\bar{z} = f(\bar{x}, \bar{y})$, where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are bit strings of lengths n_x, n_y, and n_z, respectively, and f is a Boolean function computed by a polynomial size Boolean circuit C_f with $n_x + n_y$ input wires and n_z output wires.

In a *private evaluation* of C_f, Alice furnishes the (private) input data to the first n_x input wires, and Bob furnishes the (private) input data for the remaining n_y input wires. The n_z output wires should contain the result $\bar{z} = f(\bar{x}, \bar{y})$. The corresponding functionality is

$$F(\bar{x}, \bar{y}) = (\bar{z}, \bar{z}).$$

Alice and Bob should learn nothing about each others inputs or the intermediate values of the circuit, other than what is implied by their own inputs and the output values $\bar{z}$.

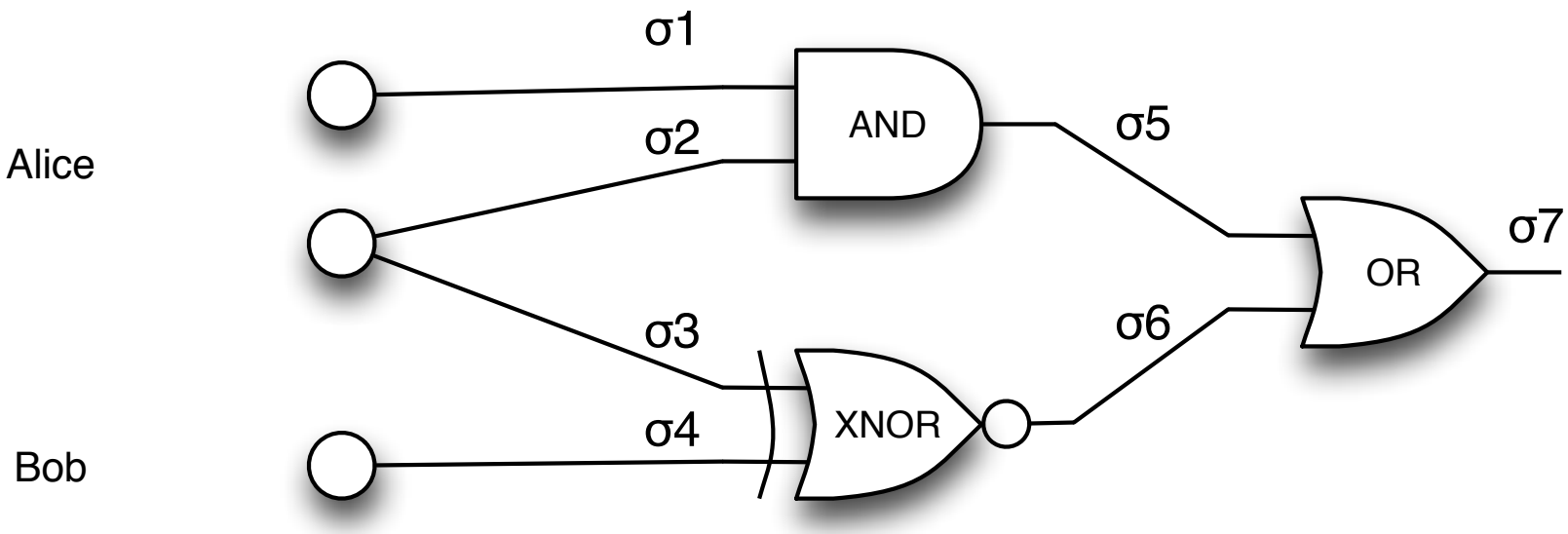
Circuit evaluation

An evaluation of a circuit assigns a Boolean value σ_w to each wire of the circuit. The input wires are assigned the corresponding input values.

Let G be a gate with input wires u and v and output wire w that computes the Boolean function $g(x, y)$. In a correct assignment, $\sigma_w = g(\sigma_u, \sigma_v)$.

A complete evaluation of the circuit first assigns values to the input wires and then works its way down the circuit, assigning a value to the output wire of any gate whose inputs have already received values.

A Boolean circuit



Private circuit evaluation

In a *private circuit evaluation*,

- Both Alice and Bob learn the output values of the circuit;
- Neither Alice nor Bob gain any information about each others private input values except for whatever is implied by their own input values and the circuit output.

We present two different schemes for privately evaluating a circuit:

- Value shares;
- Garbled circuits.