Number Theory Summary

Integers Let $\mathbf{Z}$ denote the integers and $\mathbf{Z}^+$ the positive integers.

Division For $a \in \mathbf{Z}$ and $n \in \mathbf{Z}^+$, there exist unique integers q, r such that $a = nq + r$ and $0 \leq r < n$. We denote the *quotient* q by $\lfloor a/n \rfloor$ and the *remainder* r by $a \bmod n$. We say n *divides* a (written $n \,|\, a$) if $a \bmod n = 0$. If $n \,|\, a$, n is called a *divisor* of a. If also $1 < n < |a|$, n is said to be a *proper divisor* of a.

Greatest common divisor The *greatest common divisor* (gcd) of integers a, b (written $\gcd(a, b)$ or simply (a, b)) is the greatest integer d such that $d \,|\, a$ and $d \,|\, b$. If $\gcd(a, b) = 1$, then a and b are said to be *relatively prime*.

Euclidean algorithm Computes $\gcd(a, b)$. Based on two facts: $\gcd(0, b) = b$; $\gcd(a, b) = \gcd(b, a - qb)$ for any $q \in \mathbf{Z}$. For rapid convergence, take $q = \lfloor a/b \rfloor$, in which case $a - qb = a \bmod b$.

Congruence For $a, b \in \mathbf{Z}$ and $n \in \mathbf{Z}^+$, we write $a \equiv b \pmod{n}$ iff $n \mid (b - a)$. Note $a \equiv b \pmod{n}$ iff $(a \bmod n) = (b \bmod n)$.

Modular arithmetic Fix $n \in \mathbf{Z}^+$. Let $\mathbf{Z}_n = \{0, 1, \ldots, n - 1\}$ and let $\mathbf{Z}_n^* = \{a \in \mathbf{Z}_n \mid \gcd(a, n) = 1\}$. For integers a, b, define $a \oplus b = (a + b) \bmod n$ and $a \otimes b = ab \bmod n$. $\oplus$ and $\otimes$ are associative and commutative, and $\otimes$ distributes over $\oplus$. Moreover, $\bmod n$ distributes over both $+$ and $\times$, so for example, $a + b \times (c + d) \bmod n = (a \bmod n) + (b \bmod n) \times ((c \bmod n) + (d \bmod n)) = a \oplus b \otimes (c \oplus d)$. $\mathbf{Z}_n$ is closed under $\oplus$ and $\otimes$, and $\mathbf{Z}_n^*$ is closed under $\otimes$.

Primes and prime factorization A number $p \geq 2$ is *prime* if it has no proper divisors. Any positive number n can be written uniquely (up to the order of the factors) as a product of primes. Equivalently, there exist unique integers $k, p_1, \ldots, p_k, e_1, \ldots, e_k$ such that $n = \prod_{i=1}^{k} {p_i}^{e_i}$, $k \geq 0$, $p_1 < p_2 < \ldots < p_k$ are primes, and each $e_i \geq 1$. The product $\prod_{i=1}^{k} {p_i}^{e_i}$ is called the *prime factorization* of n. A positive number n is *composite* if $(\sum_{i=1}^{k} e_i) \geq 2$ in its prime factorization. By these definitions, $n = 1$ has prime factorization with $k = 0$, so 1 is neither prime nor composite.

Linear congruences Let $a, b \in \mathbf{Z}$, $n \in \mathbf{Z}^+$. Let $d = \gcd(a, n)$. If $d \,|\, b$, then there are d solutions x in $\mathbf{Z}_n$ to the congruence equation $ax \equiv b \pmod{n}$. If $d \nmid b$, then $ax \equiv b \pmod{n}$ has no solution.

Extended Euclidean algorithm Finds one solution of $ax \equiv b \pmod{n}$, or announces that there are none. Call a triple (g, u, v) *valid* if $g = au + nv$. Algorithm generates valid triples starting with $(n, 0, 1)$ and $(a, 1, 0)$. Goal is to find valid triple (g, u, v) such that $g \mid b$. If found, then $u(b/g)$ solves $ax \equiv b \pmod{n}$. If none exists, then no solution. Given valid (g, u, v), (g', u', v'), can generate new valid triple $(g - qg', u - qu', v - qv')$ for any $q \in Z$. For rapid convergence, choose $q = \lfloor g/g' \rfloor$, and retain always last two triples. Note: Sequence of generated g-values is exactly the same as the sequence of numbers generated by the Euclidean algorithm.

Inverses Let $n \in \mathbf{Z}^+$, $a \in \mathbf{Z}$. There exists unique $b \in \mathbf{Z}$ such that $ab \equiv 1 \pmod{n}$ iff $\gcd(a, n) = 1$. Such a b, when it exists, is called an *inverse* of a modulo n. We write a^{-1} for the unique inverse of a modulo n that is also in $\mathbf{Z}_n$. Can find $a^{-1} \bmod n$ efficiently by using Extended Euclidean algorithm to solve $ax \equiv 1 \pmod{n}$.

Euler function Let $\phi(n) = |\mathbf{Z}_n^*|$. One can show that $\phi(n) = \prod_{i=1}^{k} (p_i - 1) {p_i}^{e_i - 1}$, where $\prod_{i=1}^{k} {p_i}^{e_i}$ is the prime factorization of n. In particular, if p is prime, then $\phi(p) = p - 1$, and if p, q are distinct primes, then $\phi(pq) = (p-1)(q-1)$.

Euler's theorem Let $n \in \mathbf{Z}^+$, $a \in \mathbf{Z}_n^*$. Then $a^{\phi(n)} \equiv 1 \pmod{n}$. As a consequence, if $r \equiv s \pmod{\phi(n)}$ then $a^r \equiv a^s \pmod{n}$.

Order of an element Let $n \in \mathbf{Z}^+$, $a \in \mathbf{Z}_n^*$. We define $\operatorname{ord}(a)$, the *order* of a modulo n, to be the smallest number $k \geq 1$ such that $a^k \equiv 1 \pmod{n}$. Fact: $\operatorname{ord}(a) \,|\, \phi(n)$.

Primitive roots Let $n \in \mathbf{Z}^+$, $a \in \mathbf{Z}_n^*$. a is a *primitive root* of n iff $\operatorname{ord}(a) = \phi(n)$. For a primitive root a, it follows that $\mathbf{Z}_n^* = \{a \bmod n, a^2 \bmod n, \ldots, a^{\phi(n)} \bmod n\}$. If n has a primitive root, then it has $\phi(\phi(n))$ primitive roots. Primitive roots exist for every prime p (and for some other numbers as well). a is a primitive root of p iff $a^{(p-1)/q} \not\equiv 1 \pmod{p}$ for every prime divisor q of $p - 1$.

Discrete log Let p be a prime, a a primitive root of p, $b \in \mathbf{Z}_p^*$ such that $b \equiv a^k \pmod{p}$ for some k, $0 \leq k \leq p - 2$. We say k is the *discrete logarithm* of b to the base a.

Chinese remainder theorem Let $n_1, \ldots, n_k$ be pairwise relatively prime numbers in $\mathbf{Z}^+$, let $a_1, \ldots, a_k$ be integers, and let $n = \prod_i n_i$. There exists a unique $x \in Z_n$ such that $x \equiv a_i \pmod{n_i}$ for all $1 \leq i \leq k$. To compute x, let $N_i = n/n_i$ and compute $M_i = N_i^{-1} \bmod n_i$, $1 <= i <= k$. Then $x = (\sum_{i=1}^{k} a_i M_i N_i) \bmod n$.

Quadratic residues Let $a \in \mathbf{Z}$, $n \in \mathbf{Z}^+$. a is a *quadratic residue* modulo n if there exists y such that $a \equiv y^2 \pmod{n}$. a is sometimes called a *square* and y its *square root*.

Quadratic residues modulo a prime If p is an odd prime, then every quadratic residue in $\mathbf{Z}_p^*$ has exactly two square roots in $\mathbf{Z}_p^*$, and exactly half of the elements in $\mathbf{Z}_p^*$ are quadratic residues. Let $a \in \mathbf{Z}_p^*$ be a quadratic residue. Then $a^{(p-1)/2} \equiv (y^2)^{(p-1)/2} \equiv y^{p-1} \equiv 1 \pmod{p}$, where y a square root of a modulo p. Let g be a primitive root modulo p. If $a \equiv g^k \pmod{p}$, then a is a quadratic residue modulo p iff k is even, in which case its two square roots are $g^{k/2} \bmod p$ and $-g^{k/2} \bmod p$. If $p \equiv 3 \pmod{4}$ and $a \in \mathbf{Z}_p^*$ is a quadratic residue modulo p, then $a^{(p+1)/4}$ is a square root of a, since $(a^{(p+1)/4})^2 \equiv aa^{(p-1)/2} \equiv a \pmod{p}$.

Quadratic residues modulo products of two primes If $n = pq$ for p, q distinct odd primes, then every quadratic residue in $\mathbf{Z}_n^*$ has exactly four square roots in $\mathbf{Z}_n^*$, and exactly 1/4 of the elements in $\mathbf{Z}_n^*$ are quadratic residues. An element $a \in \mathbf{Z}_n^*$ is a quadratic residue modulo n iff it is a quadratic residue modulo p and modulo q. The four square roots of a can be found from its two square roots modulo p and its two square roots modulo q using the Chinese remainder theorem.

Legendre symbol Let $a \geq 0$, p an odd prime. $\left(\frac{a}{p}\right) = 1$ if a is a quadratic residue modulo p, -1 if a is a quadratic non-residue modulo p, and 0 if $p \,|\, a$. Fact: $\left(\frac{a}{p}\right) = a^{(p-1)/2}$.

Jacobi symbol Let $a \geq 0$, n an odd positive number with prime factorization $\prod_{i=1}^{k} {p_i}^{e_i}$. We define $\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{e_i}$. (By convention, this product is 1 when $k = 0$, so $\left(\frac{a}{1}\right) = 1$.) The Jacobi and Legendre symbols agree when n is an odd prime. If $\left(\frac{a}{n}\right) = -1$ then a is definitely not a quadratic residue modulo n, but if $\left(\frac{a}{n}\right) = 1$, a might or might not be a quadratic residue.

Computing the Jacobi symbol $\left(\frac{a}{n}\right)$ can be computed efficiently by a straightforward recursive algorithm, based on the following identities: $\left(\frac{0}{1}\right) = 1$; $\left(\frac{0}{n}\right) = 0$ for $n \neq 1$; $\left(\frac{a_1}{n}\right) = \left(\frac{a_2}{n}\right)$ if $a_1 \equiv a_2 \pmod{n}$; $\left(\frac{2}{n}\right) = 1$ if $n \equiv \pm 1 \pmod{8}$; $\left(\frac{2}{n}\right) = -1$ if $n \equiv \pm 3 \pmod{8}$; $\left(\frac{2a}{n}\right) = \left(\frac{2}{n}\right)\left(\frac{a}{n}\right)$; $\left(\frac{a}{n}\right) = \left(\frac{n}{a}\right)$ if $a \equiv 1 \pmod{4}$ or $n \equiv 1 \pmod{4}$; $\left(\frac{a}{n}\right) = -\left(\frac{n}{a}\right)$ if $a \equiv n \equiv 3 \pmod{4}$.

Solovay-Strassen test for compositeness Let $n \in \mathbf{Z}^+$. If n is composite, then for roughly 1/2 of the numbers $a \in \mathbf{Z}_n^*$, $\left(\frac{a}{n}\right) \not\equiv a^{(n-1)/2} \pmod{n}$. If n is prime, then for every $a \in \mathbf{Z}_n^*$, $\left(\frac{a}{n}\right) \equiv a^{(n-1)/2} \pmod{n}$.

Miller-Rabin test for compositeness Let $n \in \mathbf{Z}^+$ and write $n-1 = 2^k m$, where m is odd. Choose $1 \leq a \leq n-1$. Compute $b_i = a^{m2^i} \bmod n$ for $i = 0, 1, \ldots, k-1$. If n is composite, then for roughly $3/4$ of the possible values for a, $b_0 \neq 1$ and $b_i \neq -1$ for $0 \leq i \leq k-1$. If n is prime, then for every a, either $b_0 = 1$ or $b_i = -1$ for some i, $0 \leq i \leq k-1$.

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