HW1, CPSC 468/568, Due Feb. 4, 2016

Problem 1 *(5 points)*:

Prove that, for any constants c and d, both greater than 1, $n^d = o(c^n)$, *i.e.*, exponential grows faster than polynomial.

Problem 2 *(9 points)*:

For each of the three lists of four functions, give an ordering f_1, f_2, f_3, f_4 such that $f_1(n) = O(f_2(n)) = O(f_3(n)) = O(f_4(n))$. You need not provide proofs, just correct orderings.

In some cases, stronger statements can be made, *i.e.*, there are pairs f_i, f_{i+1} in one or more of the correct orderings such that $f_i(n) = o(f_{i+1}(n))$. You may earn one point of extra credit for each such pair that you identify *only if you provide a proof that this little-oh relationship holds.*

1. $\log n$, $\sqrt{n}$, $(\log n)^2$, $\log\log n$.
2. $n^{4/3}$, $n\log n$, n^2, $\log(n!)$.
3. $n!$, n^n, e^n, $2^{\log n^{\log\log n}}$.

Problem 3-a *(9 points)*:

Specify a Turing Machine that, on input $\llcorner p \lrcorner$, the binary representation of a nonnegative integer p, outputs $\llcorner p+1 \lrcorner$. You may use any number of states, any alphabet, and any number of work tapes.

Example input tape:

$$\ldots 1001011111 \ldots$$

Corresponding output tape:

$$\ldots 1001100000 \ldots$$

Your solution should have running time $O(n)$, where n is the length of the input.

Problem 3-b *(7 points)*:

Provide an *oblivious* Turing Machine that computes the same function as the machine that you provided in part (a). Your solution should have running time $O(n\log n)$.

Problem 4 *(14 points)*:

Prove that `2SAT` $\in$ P.

Problem 5 *(10 points)*:

A "system of quadratic equations modulo 2 in n variables" is a set of equations over $\mathbb{Z}_2$ in which each term has degree at most 2. (For example, $x_1 \cdot x_2 + x_3 \equiv 1 \pmod 2$ is a quadratic equation modulo 2.) Such a system is said to be "solvable" if there is an assignment of values in $\mathbb{Z}_2$ to the variables $x_1, \ldots, x_n$ that satisfies all of the equations.

Prove that the language of solvable systems of quadratic equations modulo 2 is NP-complete.

Problem 6 *(13 points)*:

A graph $G(V, E)$ is k-colorable if there is an assignment f of k colors to the vertices (*i.e.*, $f : V \longrightarrow \{1, 2, \ldots, k\}$), such that, if $(a, b) \in E$, then $f(a) \neq f(b)$. That is, the endpoints of each edge are assigned different colors. Let `3Colorable` denote the set of all graphs that are 3-colorable.

Provide a many-to-one, polynomial-time reduction from `3Colorable` to `SAT`. Prove that the reduction is correct and runs in polynomial time.

Problem 7 *(10 points)*:

Prove that, if every unary NP language is in P, then EXP = NEXP.

Problem 8 *(13 points)*:

Let $\Delta = \text{NP} \cap \text{coNP}$. Prove that Δ equals the class of decision problems that are polynomial-time Cook-reducible to Δ, *i.e.*, $\Delta = \text{P}^{\Delta}$.

Problem 9 *(10 points)*:

Let $x_1 \neq x_2$ be distinct elements of $\{0, 1\}^*$. Then E_{x_1,x_2} is defined as the set of all Turing Machines M such that M does not distinguish between the input x_1 and the input x_2. (More precisely, the elements of E_{x_1,x_2} are encodings of Turing Machines as binary strings, but for brevity we will refer to them as machines.) That is, M is in E_{x_1,x_2} if M either halts on both inputs x_1 and x_2 or halts on neither, and, if it halts on both, $M(x_1) = M(x_2)$.

Prove that membership in the set E_{x_1,x_2} is not decidable.