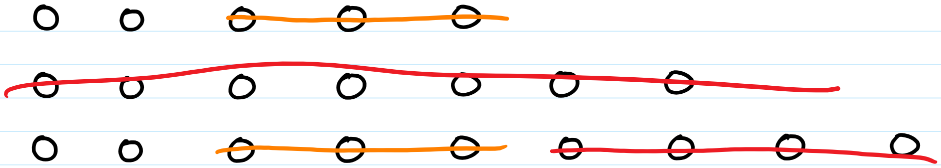
Nim

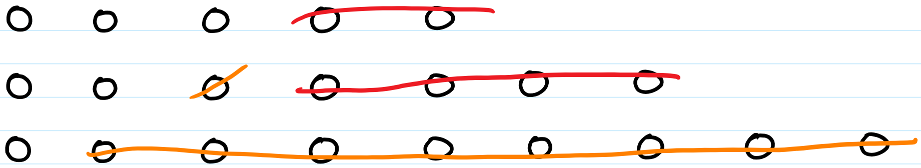


$$\begin{array}{cccc} 0&0&1&0 \\ 0&1&1&1 \\ 1&0&0&1 \\ \hline 1&1&0&0 \end{array}$$

Start with rows of $n_1, n_2, \ldots, n_k$ stones

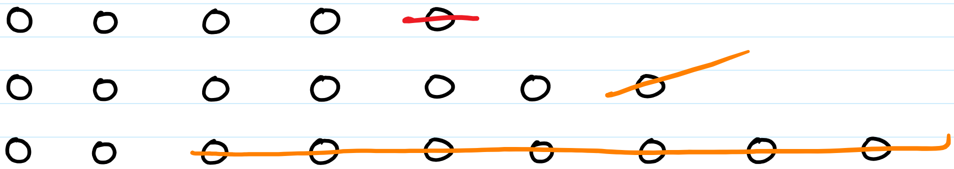
On each turn, take as many stones as you wish from one row

If no possible moves, you lose



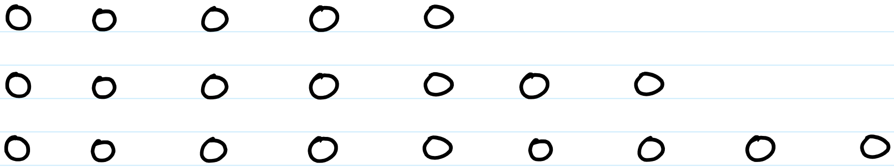
$$\begin{array}{ccc} 1&0&1 \\ 1&1&1 \\ 0&0&1 \\ \hline 0&1&1 \end{array}$$

$$\begin{array}{ccc} 1&0&1 \\ 0&1&0 \\ 0&0&1 \\ \hline 1&1&0 \end{array}$$



$$\begin{array}{ccc} 1&0&1 \\ 1&1&1 \\ 0&1&0 \\ \hline 0&0&0 \end{array}$$

$$\begin{array}{ccc} 1&0&0 \\ 1&1&0 \\ 0&1&0 \\ \hline 0&0&0 \end{array}$$



# stones in binary ↓

$$\begin{array}{lcccc} 5 & 0&1&0&1 \\ 7 & 0&1&1&1 \\ 9 & 1&0&0&1 \\ \hline & 1&0&1&1 \end{array}$$

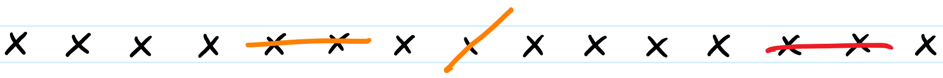
← non-zero, so there is a winning move

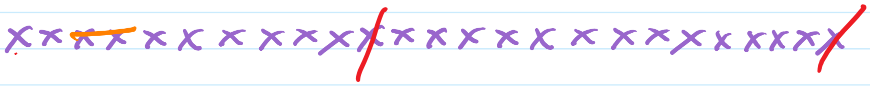
$$\begin{array}{cccc} 1&0&0&1 \\ 1&0&1&1 \\ \hline 0&0&1&0 \end{array}$$

Kayles

0 5 10 15 20 25 30 35

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

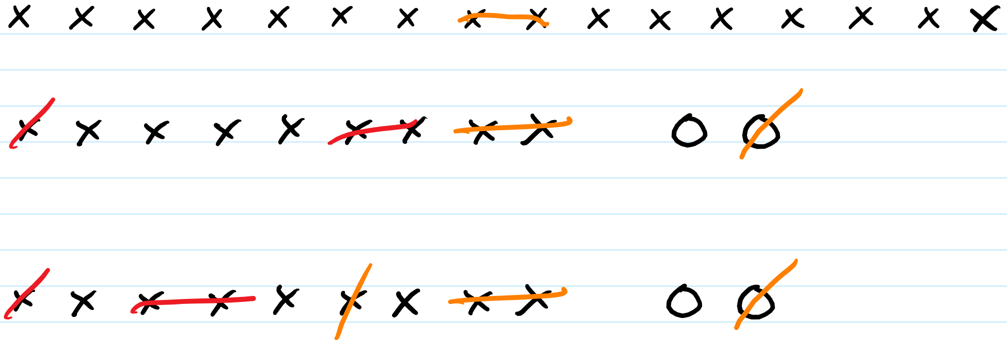




Start with row of n pins

On each turn, take 1 or 2 adjacent pins

If no possible moves, you lose



011
001
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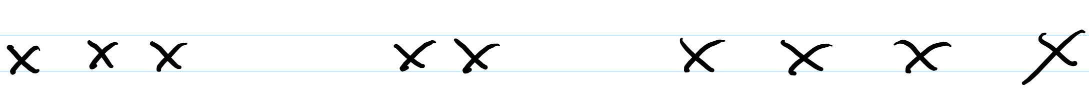


Combinatorial Games

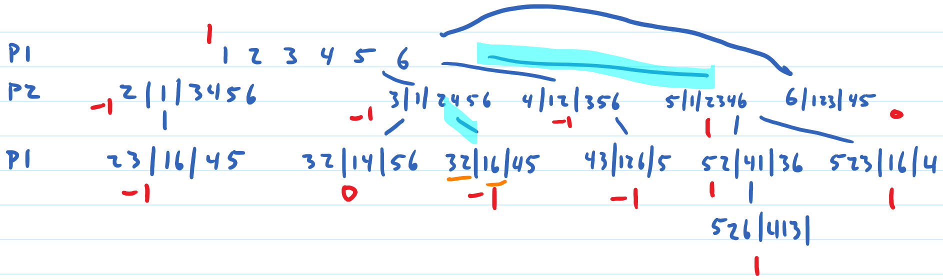
Combinatorial Game:

	Nim, Kayle's	Chess, Checkers, Go	Backgammon, Yahtzee	Poker	Roshambo	Starcraft
two-player	✓	✓	✓	✗	✓	✗
turn-based	✓	✓	✓	✓	✗	✗
non-stochastic	✓	✓	✗	✗	✓	✗
perfect information ↳ know all actions and possible outcomes for all players	✓	✓	✓	✗	✓	✗
impartial outcomes don't depend on turn	✓	✗				
normal last move wins	✓					
misere last move loses						
finite bounded # of moves						

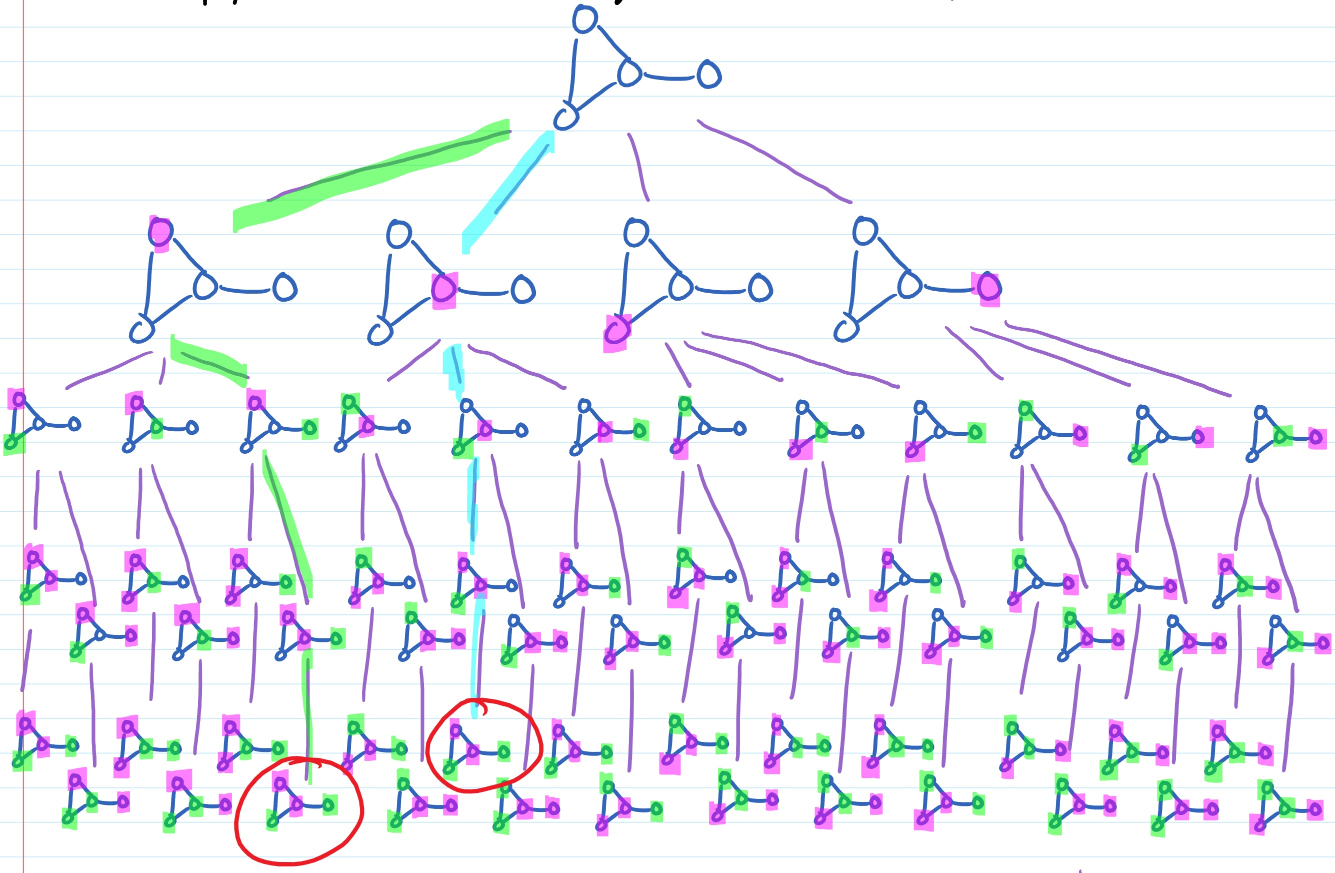
Finite Combinatorial Games

Divisors : Start with 1...n, players take turns taking a number with remaining divisors; opponent gets all the remaining divisors. Game is over when no moves remain; winner is player with higher sum (draw if =)

1 : P1 wins
0 : draw
-1 : P2 wins



Graph: take turns coloring a vertex in a graph with your color player who covers the most edges wins (draw if =)



$\#$ final positions $= \binom{4}{2} = 6$

$\frac{4!}{2!\cdot(4-2)!}$

Order positions by maximum distance to end.

Determine winner of distance 0 positions (end) by rules of game

Use recursive formula to determine value of other positions in order of increasing distance