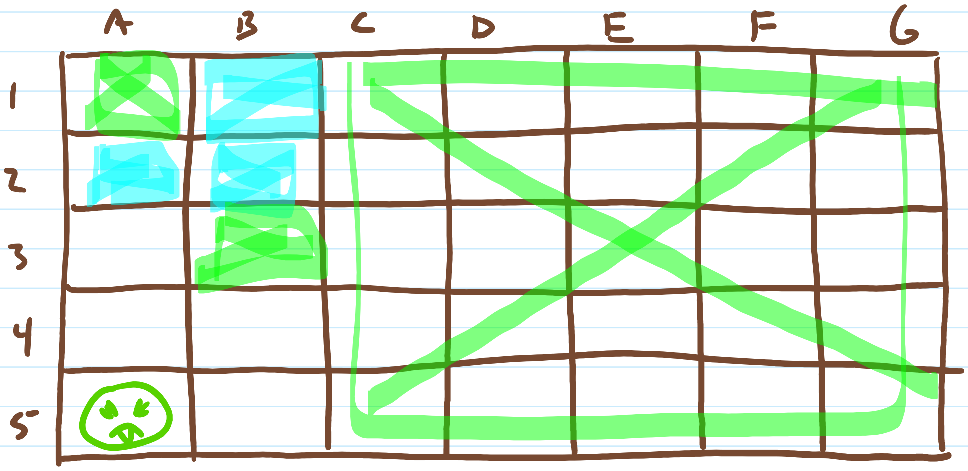
Chomp

Play on $m \times n$ grid. Take turns selecting a remaining cell, remove all above and to right.

Last move loses finite, impartial, misere



Outcome class = who has winning strategy

N	next player wins	current
P	prev player wins	

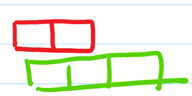
Any position in a finite, impartial, normal or misere game is an N or P position

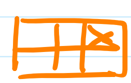
N: at least one move to P position

P: all moves to N pos

0 0 0	N –
1 0 0	P
1 1 0	N –
1 1 1	N
2 0 0	N –
2 1 0	P
2 1 1	N
2 2 0	N
2 2 1	P
2 2 2	N

increasing max # moves to end (or any order such that all moves ↑)





000
001
010
100
110
101
011
200
020
002
111
120
⋮

Kayles
. . . . .
. . . . x
. . . x .
. . . x x
⋮

For Nim, there is a winning move if and only if the bitwise exclusive or of the number of stones left in each row is non-zero, and the winning moves are the ones that make the bitwise exclusive or 0.

(So a position is an N-position if and only if exclusive-or is $\neq 0$)

O O O O
O O O O O O O
O O O O O O O O O O

Proof (strong induction on # of stones left)

Base case ($n=0$): The only game with 0 stones is already over, previous player took last stone and won, so is a P position as required.

Ind step: Suppose $k>0$ and all positions with i stones, $0 \le i < k$ are N-positions iff their Nim-sums are non-zero.
stones in each row exclusive or

[want to show it is a P pos]

Suppose position $m_1, \ldots, m_r$ has $m_1 + \cdots + m_r = k$ and $m_1 \oplus \cdots \oplus m_r = 0$. A move reduces some m_i to m_i' with $0 \le m_i' < m_i$. The Nim-sum of the result is

$$m_1 \oplus \cdots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \cdots \oplus m_r \oplus (m_i \oplus m_i)$$

$$= m_1 \oplus \cdots \oplus m_r \oplus m_i \oplus m_i' = 0 \oplus m_i \oplus m_i' \neq 0$$

since $m_i \neq m_i'$

Also, $0 \le m_1 + \cdots + m_{i-1} + m_i' + m_{i+1} + \cdots + m_r < m_1 + \cdots + m_r = k$
so ind. hyp. applies – new pos is an N pos.

since $m_i' < m_i$

Also, $0 \le m_1 + \cdots + m_{i-1} + m_i' + m_{i+1} + \cdots + m_r < m_1 + \cdots + m_r = k$
so ind. hyp. applies – new pos is an N pos.
So all moves to N pos, current P pos

Suppose position $m_1, \ldots, m_r$ has $m_1 + \cdots + m_r = k$ and
$m_1 \oplus \cdots \oplus m_r = x \ne 0$. Find most significant bit msb of x.

there must be one; otherwise that bit would not be set in x

Find i s.t. m_i has that bit set. Let $m_i' = m_i \oplus x$.
Choose move that reduces m_i to m_i'. $m_i' < m_i$, so legal!

b/c msb changed from 1 to 0

Nim-sum of result is
$$
\begin{aligned}
& m_1 \oplus \cdots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \cdots \oplus m_r \\
&= m_1 + \cdots + m_{i-1} + m_i \oplus x \oplus m_{i+1} \oplus \cdots \oplus m_r && \text{substitute } m_i' = m_i \oplus x \\
&= m_1 + \cdots + m_{i-1} + m_i \oplus m_{i+1} \oplus \cdots \oplus m_r \oplus x && \text{reorder} \\
&= x \oplus x = 0
\end{aligned}
$$

Also, $0 \le m_1 + \cdots + m_{i-1} + m_i' + m_{i+1} + \cdots + m_r < m_1 + \cdots + m_r = k$
so ind. hyp. applies – new pos is a P pos.
So there is a move to a P pos; original is an N pos.

Using Sprague-Grundy

Sprague-Grundy Theorem: every finite, normal impartial combinatorial game is equivalent to some form of 1-row Nim.

Corollary: If G is equivalent to *n and H is equivalent to *m then G+H is equivalent to $*(n \oplus m)$

For finite, normal, impartial games:

game-over position = *0
for each other position P in order of increasing length (max # moves to end)
    start with S = empty set
    for each move
        determine resulting position P'
        look up what P' is equivalent to, add to S
    compute mex(S), save that as equivalent to P

For Kayles/Nim-like games (reducing number of objects in a pile or splitting piles)

game-over position = *0 (one group)
for each other starting position P in order of increasing size
    start with S = empty set
    for each move
        determine resulting position P', write as $\underbrace{p1 + p2 + \ldots + pn}_{\text{\# objects left in each group}}$ (objects left in each pile)
        look up what each pi is equivalent to, compute exclusive-or of all; add result to S
    compute mex(S), save that as equivalent to P

Kayles	position	Reachable	mex(Reachable)	Grundy
–	K0	{ }	0	*0
x	K1	{–} ≈ {*0}	1	*1
xx	K2	{–, x} ≈ {*0, *1}	2	*2
xxx	K3	{x, xx, x·x} ≈ {*1, *2, x+x} ≈ {*1, *2, *1+*1} ≈ {*1, *2, *0}	3	*3
xxxx	K4	{xx, x··x, xxx, x·xx} ≈ {*2, *0, *3, *3}	1	*1

mex = min excludant: smallest non-neg int not in set

*0 { Number 0 = 1-row Nim w/ 0 stones

$K1 + K2 \approx *1 + *2 \approx *(1 \oplus 2) = *3$

Finding winning move

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4,
3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

xxxxx · xxxxxx · xxx · xx

*4   *3   *3   *2

$\approx *(4 \oplus 3 \oplus 3 \oplus 2) = *6$ $\begin{array}{ccc}1&1&0\\4&2&1\end{array}$

so N non-zero position

find term w/ 1 in 4's place

O O O O

O O

O O O O O O O

$4 \oplus 6 = 2$

① compute table as above up to size of largest group

② look up equivalence for each group

③ compute xor

④ find group that has a 1 in same place as MSB of xor

⑤ compute xor of that group

of xor

$4 \oplus 6 = 2$

⑤ compute xor of that group and result from ③

find a move on xxxxx that results in pos $\approx *2$

. xxxx $*1$

x. xxx $*1 + *3 \approx *(1 \oplus 3) = *2$

xxxxx xxxxxx xxxxxxxxxxxx

$\approx *3$ $\approx *2$ $\approx *6$ ← from table

$3 \oplus 2 \oplus 6 = 7$

$$\begin{array}{r} 011 \\ 010 \\ 110 \\ \hline 111 \end{array}$$

$3 \oplus 2 \oplus (6 \oplus 7)$

$(3 \oplus 2 \oplus 6) \oplus 7$
$= 0$

move on group $\approx *6$ (6, 7 have same most sig bit)

change it to $\approx *(6 \oplus 7)$
$= *1$ (so resulting $\oplus$ will be zero)

search all moves to find one that results in something $\approx *1$

.. xxxxxxxxxx
$*4$ ← from table again $4 \neq 1$

x .. xxxxxxxxx
$*1$ $*1$ $1 \oplus 1 = 0$ $0 \neq 1$

xx .. xxxxxxx
$*2$ $*2$ $2 \oplus 2 = 0$ $0 \neq 1$

winning move on group xxxxxxxxxxxx ⇒ xxx .. xxxxxx
$*3$ $*2$ $3 \oplus 2 = 1$ $1 = 1$ this is it!