Game Positions

Game position = set of options

↳ pos you can move to

In traditional 1-row Nim

		Nimbers
$\sqcup$ =	$\{\}$	$*0$
0 =	$\{\sqcup\}$	$*1$
0 0 =	$\{\sqcup, 0\}$	$*2$
0 0 0 =	$\{\sqcup, 0, 00\}$	$*3$
0 0 0 0 =	$\{\sqcup, 0, 00, 000\}$	$*4$

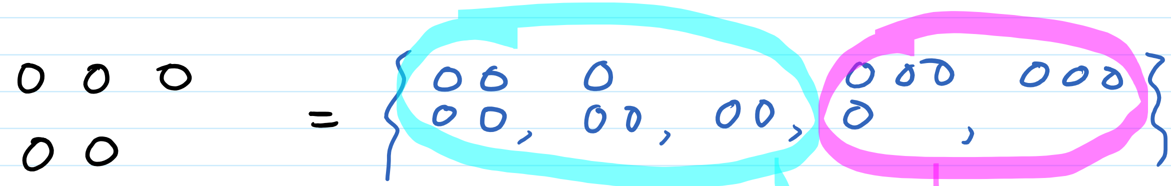
$\vdots$

Outcome class : N – next has winning strat
P – prev has winning strategy

$\{\} = *0$ is a position

Sums of Games

$G \quad *3$
$+$
$H \quad *2$



$$G + H = \{ G' + H \mid G' \text{ is an option of } G \}$$
$$\cup$$
$$\{ G + H' \mid H' \text{ is an option of } H \}$$

Equivalence of Games

For impartial, normal finite games G, G', say $G \approx G'$ if and only if for all positions H

$G+H$, $G'+H$ have same outcome class

$*****\approx *4$

Is $*2 \approx *1$?

$*2 + \underline{*2}$ P $\quad *1 + \underline{*2}$ N

$*2+*0$ N $\quad *1+*0$ N

$*2+*5$ N $\quad *1+*5$ N

Is $*5 \approx *3$?

$*5 + \underline{*3}$ N $\quad *3 + \underline{*3}$ P

$*3+*3$ is an option

Conjecture: $\forall m,n \in \mathbb{N}$, $m \neq n \to \quad *m \not\approx *n$

Is $*2 + *1 \approx *3$

$*2+*1+\underline{*0}$ N $\quad *3+\underline{*0}$ N

$*2+*1+\underline{*1}$ N $\quad *3+\underline{*1}$ N

$*2+*1+\underline{*2}$ N $\quad *3+\underline{*2}$ N

$*2+*1+\underline{*3}$ P $\quad *3+\underline{*3}$ P

$*2+*1+\underline{*4}$ N $\quad *3+\underline{*4}$ N

Conjecture: $*n + *m \approx$

Properties of Equivalence

For all finite, impartial, normal games G, H, K

$G \approx H \rightarrow$ G, H have same outcome class

$G \approx G$ reflexive

$G \approx H \longleftrightarrow H \approx G$ symmetric

$G \approx H$ and $H \approx K \rightarrow G \approx K$ transitive

equivalence relation

$G+H \approx H+G$

$(G+H)+K \approx G+(H+K)$

$G+K$, $H+K$ same outcome for all K

$G \;\; H$

$G+0$, $H+0$ have same outcome class

Lemmas

L1: Any position $G+H$ is an N position if G, H are in different outcome classes and is a P position if G, H are both P positions.

Proof: (Induction on length of game: $\forall n \geq 0$, $\forall$ pos $G+H$ of length n, ...)

Base case: $(n=0)$ Then $\underset{P}{G+H = \{\,\}}$ so $\underset{P}{G=\{\,\}}$ $\underset{P}{H=\{\,\}}$

Ind step: Suppose $G+H$ has length $k>0$ and suppose sums of length $<k$ satisfy

3 cases 1) G in N, H is P (want $G+H$ is N pos ↳ can move to P)

$\exists$ opt G' of G s.t. G' is P pos (def N)

$G'+H$ has length $<k$, so ind. hyp. applies: $P+P$ is P

$G+H$ has opt $G'+H$ that is P, so $G+H$ is N

2) G is P, H is N symmetric w/ 1

3) G, H both P

So every G' is N and every H' is N (def P)

Ind hyp applies to each $G'+H$ or $G+H'$: all are $N+P$ or $P+N$, so all are N

So $G+H$ is P

L2: For every P position A and every position G, $G+A \approx G$

Proof: Suppose A is a P position and G is any position

Let H be any pos. [want $G+A+H$, $G+H$ same outcome class]

Two cases: $G+H$ is P $\qquad G+A+H \approx \underbrace{G+H}_{P}+\underbrace{A}_{P}$

P position (L1)

$G+H$ is N $\qquad G+A+H \approx \underbrace{G+H}_{N}+\underbrace{A}_{P}$

N pos (L1)

L3: $G \approx G'$ if and only if $G+G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$. Then $G+G$, $G+G'$ have same outcome class
$G+G$ is P (mirror)
so $G+G'$ is P too

$\leftarrow$: Suppose $G+G'$ is a P position

Then $G+(G+G') \approx G$ (L2)

and $G'+(G+G) \approx G'$ (L2)

associative, commutative

so $G \approx G+(G+G') \approx (G+G)+G' \approx G'+(G+G) \approx G'$

so $G \approx G'$ (transitive)

L4: If $G = \{G_1, \ldots, G_\ell\}$ and $G_1 \approx *n_1$ and $\cdots$ and $G_\ell \approx *n_\ell$
then $G \approx \{*n_1, \ldots, *n_\ell\}$

[use L3: show $G + \{*n_1, \ldots, *n_\ell\}$ is P pos]

Consider options of $G+\{*n_1, \ldots, *n_\ell\}$

1) $\underbrace{G_i + \{*n_1, \ldots, *n_\ell\}}$, (move on G to one of $G_1, \ldots, G_\ell$)

N pos b/c has option $G_i + *n_i$ where $G_i \approx *n_i$, which is a P pos (L3)

2) $\underbrace{G + *n_i}$, (move on $\{*n_1, \ldots, *n_\ell\}$)

N pos b/c has option $G_i + *n_i$, which is a P pos

All options are N pos, so $G+\{*n_1, \ldots, *n_\ell\}$ is a P pos

so $G \approx \{*n_1, \ldots, *n_\ell\}$ (L3)

Every finite, impartial normal game is equivalent to some nimber.

Proof :

Base case ($n=0$): only game with length 0 is $\{\} = *0 \approx *0$

Induction step: Let G be a game of length $k>0$ and suppose all games G' of length $<k$ are equivalent to some nimber.

Write $G = \{G_1, \dots, G_\ell\}$ $\quad G_1, \dots, G_\ell$ have len $<k$

So by induction hypothesis, can find $n_1, \dots, n_\ell$ s.t. $G_1 \approx *n_1, \dots, G_\ell \approx *n_\ell$

So $G \approx \underbrace{\{*n_1, \dots, *n_\ell\}}_{G'}$, $\quad$ (L4)

Claim: $G' + *m$ is P-pos where $m = \text{mex}(\{n_1, \dots, n_\ell\})$
so $G' \approx *m$ $\quad$ (L3)

[will show $G' + *m$ is P by showing all options are N]

Consider all options of $G' + *m$
Three cases: i) $G' + *j$, $j < m$ (move on $*m$)

$*j$ is an option of G' $\quad$ (mex)

so $\underbrace{*j + *j}_{P}$ is an option of $\underbrace{G' + *j}_{N}$

ii) $*i + *m$, $\quad i < m$
$\underbrace{*i + *i}_{P}$ is option of $\underbrace{*i + *m}_{N}$

iii) $*i + *m$, $\quad i > m$
$\underbrace{*m + *m}_{P}$ is opt of $\underbrace{*i + *m}_{N}$

move on G' to some $*i \in \{*n_1, \dots, *n_\ell\}$

iv) ~~$*i + *m$~~, $i = m$ $\quad$ ruled out because m is mex of

All options of $G' + *m$ are N-positions
So $G' + *m$ is P

$\therefore G' \approx *m$ $\quad$ (L3)

$G' \approx G$

so $G \approx *m$ $\quad$ (trans)

Theorem : $*n + *m \approx *(n \oplus m)$

Proof: (induction on length of game, $n+m$)

Base case ($n+m=0$): Then $n=0$, $m=0$, $n\oplus m = 0$

$*n + *m = *0 + *0 = \{\} = *0$

Induction Step: Suppose $n+m > 0$ and all n', m' s.t. $n'+m' \le n+m$ have $*n' + *m' \approx *(n' \oplus m')$

$$*n + *m = \{*0 + *m, \dots, *(n-1) + *m, *n + *0, \dots, *n + *(m-1)\}$$

$$\approx \{*(0\oplus m), \dots, *((n-1)\oplus m), *(n\oplus 0), \dots, *(n\oplus(m-1))\}$$ (ind. hyp, L4)

$$\approx *\mathrm{mex}(\{0\oplus m, \dots, (n-1)\oplus m, n\oplus 0, \dots, n\oplus(m-1)\})$$ (Sprague - Grundy)

Claim: $\mathrm{mex}(\{0\oplus m, \dots, (n-1)\oplus m, n\oplus 0, \dots, n\oplus(m-1)\}) = n\oplus m$

1) $n\oplus m$ is excluded : suppose $n\oplus m = i\oplus m$, $i<n$

then $n\oplus m\oplus m = i\oplus m\oplus m$

$n = i$ $\Rightarrow\Leftarrow$

suppose $n\oplus m = n\oplus i$, $i<m$

then $n\oplus n\oplus m = n\oplus n\oplus i$

$m = i$ $\Rightarrow\Leftarrow$

2) All x s.t. $0 \le x < n\oplus m$ are included:

Find most significant bit where $x, n\oplus m$ differ

That bit is 1 in $n\oplus m$ and 0 in x (x is smaller)

To be 1 in $n\oplus m$, corresponding bits in n, m are 0,1 or 1,0

n	$d_1\ d_2 \dots 1 \dots$
m	$e_1\ e_2 \dots 0 \dots$
$n\oplus m$	$a_1\ a_2 \dots 1 \dots b_k$
x	$a_1\ a_2 \dots 0 \dots c_k$
$m\oplus x$	$d_1\ d_2 \dots 0 \dots$

$e_1 \oplus a_1$
$= e_1 \oplus d_1 \oplus e_1$
$= d_1$

Assume, WLOG, bits are 1 in n, 0 in m

So $m\oplus x < n$

and $*(m\oplus x) + *m$ is an option of $*n + *m$

But $*(m\oplus x) + *m \approx *(m\oplus x\oplus m)$ (ind. hyp.)

$= *x$

So $*x$ is included in