

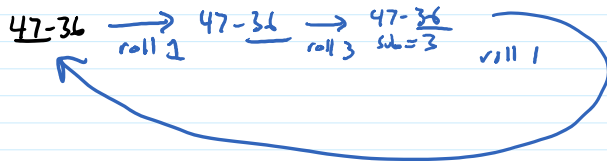
2-players, turn-based

On each turn

- roll
- if 1, then turn over
- else add number to turn total
- decide: repeat
- stop (and add turn total to score)

1st to 100 points wins

$$E[x, y] = P(\text{next player wins, given that score is } x \text{ to } y)$$



cycle \rightarrow game is infinite
can't compute by plain backwards induction

Make the game finite: introduce a turn limit

\rightarrow for high enough ^{even} turn limit, $P(\text{reach turn limit}) \approx 0$ and $E[x, y] \approx E[x, y, n]$

$E[x, y, n]$ = expected # wins for P1 given score is x to y w/ n turns left

$$E[x, y, n] = \begin{cases} 1.0 & \text{if } x \geq T \text{ and } y < T \\ 0.0 & \text{if } y \geq T \text{ and } x < T \\ 0.5 & \text{if } n = 0 \\ E_{xyn}[0] & \text{otherwise} \end{cases}$$

even $n \rightarrow$ P1's turn
odd $n \rightarrow$ P2's turn

$E[x, y]$ depends on $E[y, x]$,
 $E[y, x]$ depends on $E[x, y]$

so $E[x, y]$, $E[y, x]$ are fns of themselves

value iteration: guess values
perform calculations
update guesses
perform calculations
update guesses
repeat until converged

$E_{xyn}(t)$ = expected wins for P1 when score is x to y , n turns left, turn total = t

$$E_{xyn}(t) = \begin{cases} 1.0 & \text{if } n \text{ even and } t+x \geq T \\ 0.0 & \text{if } n \text{ odd and } t+y \geq T \\ \max\left(\underbrace{E[x+t, y, n-1]}_{\text{value of stopping}}, \underbrace{\frac{1}{6} \cdot E[x, y, n-1]}_{\text{roll = 1, lose progress}} + \frac{1}{6} \sum_{r=2}^6 E_{xyn}(t+r) \right) & \text{if } n \text{ even} \\ \min\left(E[x, y+t, n-1], \frac{1}{6} \cdot E[x, y, n-1] + \frac{1}{6} \sum_{r=2}^6 E_{xyn}(t+r) \right) & \text{if } n \text{ odd} \end{cases}$$

initialize $E[x, y, 0]$ according to base cases of recurrence

$n \leftarrow 0$
repeat

initialize $E[x, y, 0]$ according to base cases of recurrence

$n \leftarrow 0$
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for $x \leftarrow 0$ to T
for $y \leftarrow 0$ to T

$$E_{xy}[t] \leftarrow \begin{cases} 1.0 & \text{for } t \geq T-x \text{ if } n \text{ even} \\ 0.0 & \text{for } t \geq T-y \text{ if } n \text{ odd} \end{cases}$$

for $t = \begin{cases} T-x-1 & \text{if } n \text{ even} \\ T-y-1 & \text{if } n \text{ odd} \end{cases}$ down to 0

$$E_{xy}[t] \leftarrow \text{value from recurrence}$$

$$E[x, y, n] \leftarrow E_{xy}[0]$$

$n \leftarrow n+1$
until n odd and $n \geq 3$ and $E[x, y, n-1]$ close enough to $E[x, y, n-3]$ for all x, y

OR (value iteration on each pair of entries separately)

$E[x, y]$ = expected wins for next player when score is $x-y$

could have $E_1[x, y]$ = expected wins for P_1 when score is $x-y$, and P_1 is next

$E_2[x, y]$ = expected wins for P_2 when score is $x-y$, and P_2 is next

but then $E_2[x, y] = 1 - E_1[x, y]$, so only need E_1 (and just call it E)

$$E[T, y] \leftarrow 1.0 \text{ for all } y$$

$$E[x, T] \leftarrow 0.0 \text{ for all } x$$

$E[x, y], E[y, x]$ depend on each other, so compute both at same time

for (x, y) in $\{0 \dots T\} \times \{0 \dots T\}$ s.t. $x \geq y$ in order of $\downarrow x+y$

repeat

$$E_{xy}[t] \leftarrow 1.0 \text{ for } t \geq T-x$$

$$E_{yx}[t] \leftarrow 1.0 \text{ for } t \geq T-y$$

for $t \leftarrow T-x-1$ down to 0

$$E_{xy}[t] \leftarrow \max(1.0 - E_{yx}[t], \frac{1}{2}(1.0 - E_{yx}[t] + \sum_{r=2}^6 E_{xy}[t+r]))$$

account for change of turn

for $t \leftarrow T-y-1$ down to 0

$$E_{yx}[t] \leftarrow \max(1.0 - E_{xy}[t], \frac{1}{2}(1.0 - E_{xy}[t] + \sum_{r=2}^6 E_{yx}[t+r]))$$

$$E[x, y] \leftarrow E_{xy}[0]$$

$$E[y, x] \leftarrow E_{yx}[0]$$

until $E_{xy}[0]$ and $E_{yx}[0]$ have converged

