2-players, turn-based

On each turn
- roll
- if 1, then turn over
- else add number to turn total
- decide: repeat
  - stop (and add turn total to score)

1st to 100 points wins

$E[x,y] = P(\text{next player wins, given that score is } x \text{ to } y)$

$\underline{47}\text{-}36 \xrightarrow[\text{roll 2}]{} 47\text{-}\underline{36} \xrightarrow[\text{roll 3}]{} 47\text{-}\underline{36}$ sub = 3, roll 1 $\to \underline{47}\text{-}36$

cycle → game is infinite
can't compute by plain backwards induction

Make the game finite: introduce a turn limit

for high enough, even turn limit, $P(\text{reach turn limit}) \approx 0$ and $E[x,y] \approx E[x,y,n]$

$E[x,y,n]$ = expected # wins for P1 given score is x to y w/ n turns left

even n → P1's turn
odd n → P2's turn

$$= \begin{cases} 1.0 & \text{if } x \geq T \text{ and } y < T \\ 0.0 & \text{if } y \geq T \text{ and } x < T \\ 0.5 & \text{if } n = 0 \\ E_{xyn}[0] & \text{otherwise} \end{cases}$$

$E[x,y]$ depends on $E[y,x]$,
$E[y,x]$ depends on $E[x,y]$

so $E[x,y]$, $E[y,x]$ are fns of themselves

value iteration: guess values

repeat until converged:
- perform calculations
- update guesses
- perform calculations
- update guesses
- ⋮

$E_{xyn}[t]$ = expected wins for P1 when score is x to y, n turns left, turn total $= t$

$$= \begin{cases} 1.0 & \text{if } n \text{ even and } t+x \geq T \\ 0.0 & \text{if } n \text{ odd and } t+y \geq T \\ \max\Big(\underbrace{E[x+t,y,n-1]}_{\text{value of stopping}}, \underbrace{\tfrac{1}{6}\cdot E[x,y,n-1]}_{\text{roll a 1, lose progress}} + \tfrac{1}{6}\sum_{r=2}^{6} E_{xyn}[t+r]\Big) & \text{if } n \text{ even} \\ \min\Big(E[x,y+t,n-1], \tfrac{1}{6}\cdot E[x,y,n-1] + \tfrac{1}{6}\sum_{r=2}^{6} E_{xyn}[t+r]\Big) & \text{if } n \text{ odd} \end{cases}$$

initialize $E[x,y,0]$ according to base cases of recurrence

$n \leftarrow 0$
repeat

initialize $E[x,y,0]$ according to base cases of recurrence

$n \leftarrow 0$
repeat
    for $x \leftarrow 0$ to T
        for $y \leftarrow 0$ to T
            $E_{xyn}[t] \leftarrow \begin{cases} 1.0 \text{ for } t \geq T-x \text{ if } n \text{ even} \\ 0.0 \text{ for } t \geq T-y \text{ if } n \text{ odd} \end{cases}$
            for $t = \begin{cases} T-x-1 \text{ if } n \text{ even} \\ T-y-1 \text{ if } n \text{ odd} \end{cases}$ downto 0
                $E_{xyn}[t] \longleftarrow$ value from recurrence
            $E[x,y,n] \leftarrow E_{xyn}[0]$
    $n \leftarrow n+1$
until n odd and $n \geq 3$ and $E[x,y,n-1]$ close enough to $E[x,y,n-3]$ for all x,y

OR (value iteration on each pair of entries separately)

$E[x,y]$ = expected wins for next player when score is $x-y$
could have $E_1[x,y]$ = expected wins for P1 when score is $x-y$, and P_1 is next
$E_2[x,y]$ = expected wins for P1 when score is $x-y$, and P_2 is next
but then $E_2[x,y] = 1 - E_1[y,x]$, so only need E_1
(and just call it E)

$E[T,y] \leftarrow 1.0$ for all y
$E[x,T] \leftarrow 0.0$ for all x

$E(x,y)$, $E(y,x)$ depend on each other,
so compute both at same time

for (x,y) in $\{0\ldots T\} \times \{0 \ldots T\}$ s.t. $x \geq y$ in order of $\downarrow$ $x+y$

$E[x,y]$
$E[y,x]$

    repeat
        $E_{xy}[t] \leftarrow 1.0$ for $t \geq T-x$
        $E_{yx}[t] \leftarrow 1.0$ for $t \geq T-y$
        account for change of turn
        for $t \leftarrow T-x-1$ downto 0
            $E_{xy}[t] \longleftarrow \max\left(1.0 - E[y,x+t],\ \frac{1}{6}\left(1.0 - E[y,x] + \sum_{r=2}^{6} E_{xy}[t+r]\right)\right)$

        for $t \leftarrow T-y-1$ downto 0
            $E_{yx}[t] \longleftarrow \max\left(1.0 - E[x,y+t],\ \frac{1}{6}\left(1.0 - E[x,y] + \sum_{r=2}^{6} E_{yx}[t+r]\right)\right)$
        $E[x,y] \leftarrow E_{xy}[0]$
        $E[y,x] \leftarrow E_{yx}[0]$
    until $E_{xy}[0]$ and $E_{yx}[0]$ have converged