Simultaneous Play Games

	W	X	Y	Z	
A	$\frac{2}{3}$	$\boxed{\frac{1}{2}}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$
B	-1	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$	-1
C	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	-1
	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$v^- = \frac{1}{2}$

$v^+ = \frac{1}{2}$

	R	P	S	
Rock	0	-1	1	-1
Paper	1	0	-1	-1
Scissors	-1	1	0	-1
	1	1	1	$v^- = -1$

$v^+ = 1$

v^- : min amount guaranteed to I
$= \max_i \min_j a_{ij}$

v^+ : ceiling on what II loses
$= \min_j \max_i a_{ij}$

For any constant-sum game, $v^- \leq v^+$

Saddle point : where neither player has incentive to unilaterally change strategy

i^*, j^* is a saddle point means $a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}$

(i^*j^* is min in its row, max in its column)

each player always chooses a single row/col

A constant-sum game A has a saddle point in <u>pure strategies</u> if and only if $v^- = v^+$

$\Rightarrow$: Suppose A has a saddle point in pure strategies.

Then $\exists\ i^*, j^*$ s.t. $\forall\ i, j$ $\quad a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}$

$$\begin{array}{cc} \begin{pmatrix} 5 & 6 & 7 \\ 3 & 1 & 8 \\ 2 & 8 & 1 \end{pmatrix} & \begin{matrix} 5 \\ 1 \\ 1 \end{matrix} \\ \begin{matrix} 5 \quad & 8 \quad & 8 \end{matrix} & v^- = 5 \end{array}$$

$v^+ = 5$

$\Leftarrow$ Suppose $v^- = v^+$. Let i^* be the i s.t. $v^- = \max_i \min_j a_{ij}$
j^* be the j s.t. $v^+ = \min_j \max_i a_{ij}$

Suppose there are 2 saddle points in pure strategies (i_1, j_1) and (i_2, j_2)

with values $a_{i_1 j_1} = x$ $\quad a_{i_2 j_2} = y$

$$
\begin{array}{c c}
 & \begin{array}{cc} j_1 & j_2 \end{array} \\
\begin{array}{c} i_1 \\ i_2 \end{array} & \begin{pmatrix} x & b \\ a & y \end{pmatrix}
\end{array}
$$

$y \le a \le x \le b \le y$

but $y = y$ (not $y < y$)

$\therefore \ y = a = x = b = y$

Mixed Strategies – probability distribution over all options

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$X = (x_1, \ldots, x_n)$ I plays row i w/ prob x_i

$Y = (y_1, \ldots, y_m)$ II plays col j w/ prob y_j

$$E(X,Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \cdot P(\text{I plays row } i \text{ and II plays col } j)$$

$$= \sum \sum a_{ij} \cdot P(\text{I plays } i) \cdot P(\text{II plays } j)$$

$$= \sum \sum a_{ij} \cdot x_i \cdot y_j$$

$$= X \cdot A \cdot Y^T$$

$X^* = \left(\tfrac{1}{3} \ \tfrac{1}{3} \ \tfrac{1}{3}\right) = Y^*$

$$E(X^*, Y^*) = \left(\tfrac{1}{3} \ \tfrac{1}{3} \ \tfrac{1}{3}\right) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ \tfrac{1}{3} \end{pmatrix}$$

$$(0 \ \ 0 \ \ 0) \begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ \tfrac{1}{3} \end{pmatrix} = 0$$

X^*, Y^* is a saddle point in mixed strategies means

$$E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y) \quad \text{for all } X, Y$$

Minimax Theorem : Every zero-sum game has a saddle point in mixed strategies

Equilibrium Theorem: If X^*, Y^* is a saddle point in mixed strategies for A with $x_i > 0$ $y_j > 0$ then

$$E(X^*, j) = E(i, Y^*) = \underline{\text{value}(A)} = E(X^*, Y^*)$$

Suppose $X^* = \left(\tfrac{1}{2} \ \tfrac{1}{4} \ \tfrac{1}{4}\right)$ and $Y^* = \left(\tfrac{2}{3} \ \tfrac{1}{3} \ 0\right)$ then

$$E(X^*, Y^*) = E(\overset{(1\ 0\ 0)}{1}, Y^*) = E(\overset{(0\ 1\ 0)}{2}, Y^*) = E(\overset{(0\ 0\ 1)}{3}, Y^*)$$

$$E(X^*, \underset{(1\ 0\ 0)}{1}) = E(X^*, \underset{(0\ 1\ 0)}{2})$$

$Y^* = \left(\frac{2}{3}\ \frac{1}{3}\ 0\right)$ then $E(X^*, \underbrace{1}_{(1\,0\,0)}) = E(X^*, \underbrace{2}_{(0\,1\,0)})$

but possibly $E(X^*, 3) > E(X^*, Y^*)$

Best Response: Best response to a mixed strategy X is Y that minimizes XAY^T $(= E(X,Y))$
" " Y is X that maximizes "

Strategies in saddle point are best responses to each other
(can't have $E(X^*, Y^*) > E(X^*, Y)$
or $E(X^*, Y^*) < E(X, Y^*)$)

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \qquad X = \left(\frac{3}{10}\ \frac{1}{5}\ \frac{1}{2}\right)$$

best response is $Y = (y_1\ y_2\ y_3)$

to maximize $y_1 \cdot E(X,1) + y_2 \cdot E(X,2) + y_3 \cdot E(X,3)$ (w/ $0 \le y_1, y_2, y_3 \le 1$, $y_1 + y_2 + y_3 = 1$)

$= -\frac{3}{10} y_1 - \frac{1}{5} y_2 + \frac{1}{10} y_3$

$y_1 = 1 \quad y_2 = 0 \quad y_3 = 0$

$Y = (1\ 0\ 0)$

Finding Saddle Points in Mixed Strategies

Thm: X^*, Y^* is a saddle point in mixed strategies and $\text{value}(A) = E(X^*, Y^*)$ if and only if

$$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j) \quad \text{for all } i, j$$

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$X^* = (\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}) = Y^*$

$E(X^*, Y^*) = 0$

$$\begin{aligned} E(1, Y^*) &= 0 \overset{?}{\le} 0 \\ E(2, Y^*) &= 0 \le 0 \\ E(3, Y^*) &= 0 \le 0 \end{aligned}$$

$$\begin{aligned} E(X^*, 1) &= 0 \ge 0 \\ E(X^*, 2) &= 0 \ge 0 \\ E(X^*, 3) &= 0 \ge 0 \end{aligned}$$

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Claim: $X^* = (\tfrac{2}{7}, 0, \tfrac{5}{7})$, $Y^* = (\tfrac{5}{7}, \tfrac{2}{7}, 0)$ is saddle pt with $E(X^*, Y^*) = \tfrac{2}{7}$

Is
$$\begin{aligned} E(X^*, 1) &= \tfrac{2}{7} \cdot .30 + \tfrac{5}{7} \cdot .28 = \tfrac{200}{700} \ge \tfrac{2}{7} \ \checkmark \\ E(X^*, 2) &= \tfrac{2}{7} \cdot .25 + \tfrac{5}{7} \cdot .30 = \tfrac{200}{700} \ge \tfrac{2}{7} \ \checkmark \\ E(X^*, 3) &= \tfrac{2}{7} \cdot .2 + \tfrac{5}{7} \cdot .33 = \tfrac{205}{700} \ge \tfrac{2}{7} \ \checkmark \end{aligned}$$

Is
$$\begin{aligned} E(1, Y^*) &= \tfrac{2}{7} \le \tfrac{2}{7} \ \checkmark \\ E(2, Y^*) &= \tfrac{196}{700} \le \tfrac{2}{7} \ \checkmark \\ E(3, Y^*) &= \tfrac{2}{7} \le \tfrac{2}{7} \ \checkmark \end{aligned}$$

So X^*, Y^* is a saddle point

Suppose there are two saddle points in mixed strategies (X_1^*, Y_1^*) (X_2^*, Y_2^*)

$$\begin{aligned} E(X_1^*, Y_2^*) &\le E(X_2^*, Y_2^*) \\ &\le E(X_2^*, Y_1^*) \\ &\le E(X_1^*, Y_1^*) \\ &\le E(X_1^*, Y_2^*) \end{aligned}$$