NFL Strategy

downs left, yards to new downs, yards to score, ticks left

Let $E[\text{pos}]$ = P(offense wins) = P(score TD on current possession)

$$\text{Then } E[\text{pos}] = \text{value} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

expected value of wins for offense

where a_{ij} = value of running offensive play i against defensive play j

$$= \sum_{\substack{\text{outcome} \\ o}} P(o) \cdot E[\text{next}(\text{pos}, o)]$$

→ yards gained, time elapsed, turnover

fxn from old pos, outcome → new pos
(you write this fxn)

5 outcomes per off/def play combination

So need dynamic programming + linear programming

already done!
E[pos] in pickle/csv

1) populate matrix (add constant to make all positive if some are 0)
2) set up linear programs
3) display results (prob dist for offense + defense + value of game)

Finding Saddle Points in Mixed Strategies

	L	R	
L	$\frac{1}{2}$	1	$\frac{1}{2}$
R	1	$\frac{2}{3}$	$\frac{2}{3}$
	1	1	$v^- = \frac{2}{3}$
		$v^+ = 1$	

For any saddle point $X^* = (x_1 \quad 1-x_1)$
$Y^* = (y_1 \quad 1-y_1)$

$E(X^*, 1) \geq E(X^*, Y^*)$

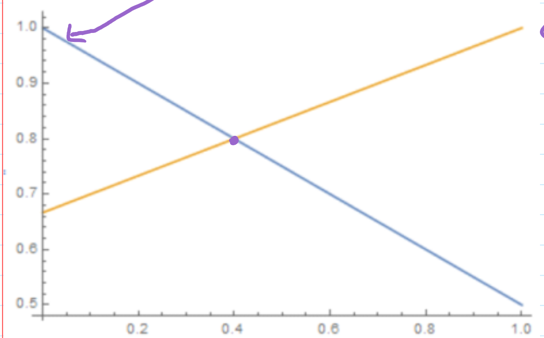
$\frac{1}{2}x_1 + 1(1-x_1) \geq v$

$\underbrace{1 - \frac{1}{2}x_1 \geq v}$

$E(X^*, 2) \geq v$

$1 \cdot x_1 + \frac{2}{3}(1-x_1) \geq v$

$\underbrace{\frac{2}{3} + \frac{1}{3}x_1 \geq v}$



for any $X = (x_1 \; 1-x_1)$, value of II's best response $= \min(E(X,1), E(X,2))$
for X^*, II's best response is Y^*; value $= \min(E(X,1), E(X,2))$
at X^*, I has no incentive to unilaterally change
only place where there is no such incentive is at the intersection

this gives X^*
(intuitively: intersection has max value $\leq E(X,1), E(X,2)$)

$1 - \frac{1}{2}x_1 = \frac{2}{3} + \frac{1}{3}x_1 \Rightarrow x_1 = \frac{2}{5}$

$X^* = (\frac{2}{5} \; \frac{3}{5})$

Repeat on other set of inequalities to get Y^*:

$E(1, Y) \leq v$
$\frac{1}{2}y_1 + 1(1-y_1) \leq v$
$1 - \frac{1}{2}y \leq v$

$1 - \frac{1}{2}y_1 = \frac{2}{3} + \frac{1}{3}y$

$y_1 = \frac{2}{5}$

$E(2, Y) \leq v$
$1 \cdot y_1 + \frac{2}{3}(1-y_1) \leq v$
$\frac{2}{3} + \frac{1}{3}y \leq v$

$Y^* = (\frac{2}{5} \; \frac{3}{5})$

(same graph b/c $A = A^T$)

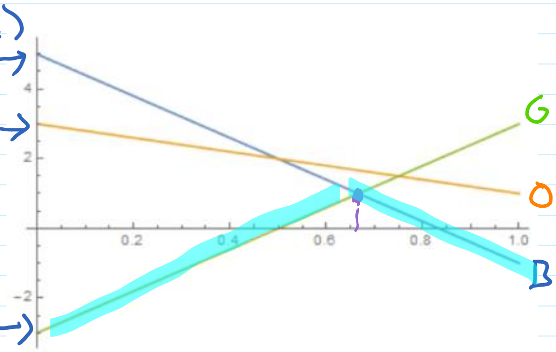
$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$

$X^* = (x_1 \quad 1-x_1)$
$Y^* = (y_1 \quad y_2 \quad 1-y_1-y_2)$

$E(X,1) = 1 \cdot x_1 + 3(1-x_1) = 3 - 2x_1 \geq v$

$E(X,2) = -1 \cdot x_1 + 5(1-x_1) = 5 - 6x_1 \geq v$

$E(X,3) = 3x_1 - 3(1-x_1) = -3 + 6x_1 \geq v$



max value $\leq$ all 3 lines is @ intersection of G, B

$-3 + 6x_1 = 5 - 6x_1 \Rightarrow x_1 = \frac{2}{3}$

$X^* = (\frac{2}{3} \; \frac{1}{3})$

Prev thm: if $Y^* = (y_1 \; y_2 \; y_3)$ and $y_j > 0$ then $E(X^*, j) = v$

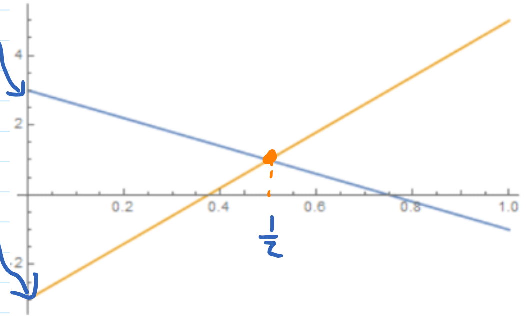
$E(X^*, 1) \neq v$, so $y_1 = 0$ ... $Y = (0 \quad y_2 \quad 1-y_2)$

$E(1,Y) = -1y_2 + 3(1-y_2) = 3-4y_2 \le v$

$E(2,Y) = 5y_2 - 3(1-y_2) = -3+8y_2 \le v$

$3-4y_2 = -3+8y_2 \Rightarrow y_2 = \frac{1}{2}$

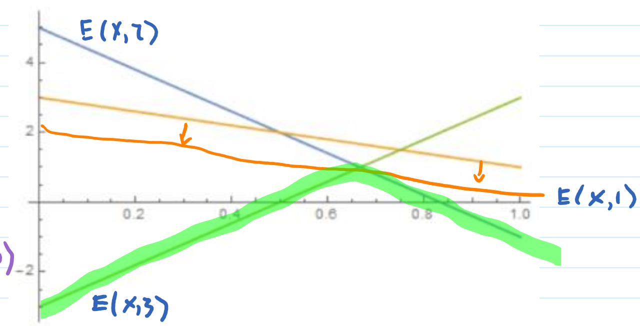
$Y^* = (0 \ \ \frac{1}{2} \ \ \frac{1}{2})$



$A = \begin{pmatrix} \boxed{0} & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$

$E(X,1)$ does not change lower envelope — can find Y^* w/ $y_1 = 0$

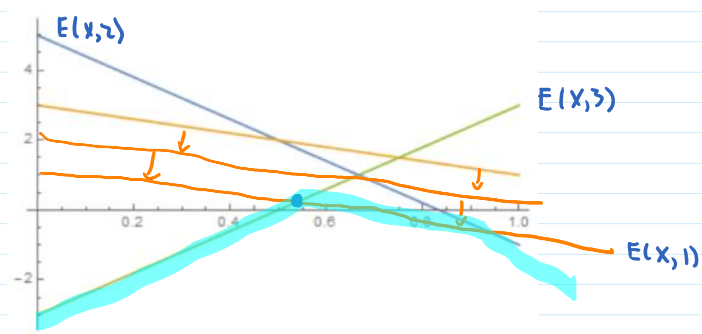
(but might be other equilibria w/ $y_1 > 0$)



$A = \begin{pmatrix} \overset{-1}{\cancel{0}} & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$

highest point on lower envelope is intersection of $E(X,1)$, $E(X,3)$

that gives X^*; Y^* has $y_2 = 0$



Linear Programming

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

original linear program →
need to standardize
(and will eliminate 1 variable)

Maximize v
subject to

$$E(X,1) = 0.30 \cdot x_1 + 0.26 \cdot x_2 + 0.28 x_3 \geq v$$

$$E(X,2) = 0.25 \cdot x_1 + 0.33 x_2 + 0.30 x_3 \geq v$$

$$E(X,3) = 0.20 \cdot x_1 + 0.28 x_2 + 0.33 x_3 \geq v$$

$$x_1 + x_2 + x_3 = 1$$

$$0 \leq x_1, x_2, x_3 \leq 1$$

$$0.2 \leq v \leq 0.33$$

divide by v, let $p_i = \frac{x_i}{v}$

must have $v \neq 0$, so make all entries positive by adding a constant (subtract out in end)

$$0.30 p_1 + 0.26 p_2 + 0.28 p_3 \geq 1$$

$$0.25 p_1 + 0.33 p_2 + 0.30 p_3 \geq 1$$

$$0.20 p_1 + 0.28 p_2 + 0.33 p_3 \geq 1$$

multiply by -1 to make $\leq$

$$-0.30 p_1 - 0.26 p_2 - 0.28 p_3 \leq -1$$

$$-0.25 p_1 - 0.33 p_2 - 0.30 p_3 \leq -1$$

$$-0.20 p_1 - 0.28 p_2 - 0.33 p_3 \leq -1$$

b_ub

al

$$0.2 \leq v \leq 0.33$$

$$0 \leq x_i \leq 1$$

$$0 \leq \frac{x_i}{v} \leq 5$$

$$0 \leq p_i \leq 5$$

bounds

c

maximize v = minimize $\frac{1}{v} = \frac{x_1 + x_2 + x_3}{v} = 1 \cdot p_1 + 1 \cdot p_2 + 1 \cdot p_3$

linear program solves for p_i, $\frac{1}{v}$; convert back to x_i, v

Java package wants $=$, not $\leq$
so add slack variables

$$-0.30 p_1 - 0.26 p_2 - 0.28 p_3 + 1 s_1 + 0 s_2 + 0 s_3 = -1$$

$$-0.25 p_1 - 0.33 p_2 - 0.30 p_3 + 0 s_1 + 1 s_2 + 0 s_3 = -1$$

$$-0.20 p_1 - 0.28 p_2 - 0.33 p_3 + 0 s_1 + 0 s_2 + 1 s_3 = -1$$

$$0 \leq s_i \leq 1$$

Use other set of inequalities to find Y^*

minimize v subject to

$$E(1,Y) = 0.30y_1 + 0.25y_2 + 0.20y_3 \le v$$
$$E(2,Y) = 0.26y_1 + 0.33y_2 + 0.28 \le v$$
$$E(3,Y) = 0.28y_1 + 0.30y_2 + 0.33 \le v$$

$y_1 + y_2 + y_3 = 1$
$y_1, y_2, y_3 \ge 0$

$\Downarrow$ divide by v, set $q_i = \frac{y_i}{v}$
(don't need to multiply by -1 b/c already $\le$)

a_z

$$0.30q_1 + 0.25q_2 + 0.20q_3 \le 1$$
$$0.26q_1 + 0.33q_2 + 0.28q_3 \le 1$$
$$0.28q_1 + 0.30q_2 + 0.33q_3 \le 1$$

bub

$$0 \le q_1 \le 5$$
$$0 \le q_2 \le 5$$
$$0 \le q_3 \le 5$$

bounds

minimize $-q_1 + -q_2 + -q_3 = -\frac{1}{v}$ (same as max $\frac{1}{v}$, same as min v)

c