Using Sprague-Grundy

Sprague-Grundy Theorem: every finite, normal, impartial combinatorial game is equivalent to some form of 1-row Nim.

Corollary: If G is equivalent to *n and H is equivalent to *m then G+H is equivalent to *(n$\oplus$m)

For finite, normal, impartial games:

game-over position = *0
for each other position P in order of increasing length (max # moves to end)
    start with S = empty set
    for each move
        determine resulting position P'
        look up what P' is equivalent to, add to S
    compute mex(S), save that as equivalent to P

```
. X X .
x . . x
x . x .
x x . .
. x x x      . . X X   . X . X   . X X .   . . . X   . X . .    -> {*2 *0 *2 *1 * 1}    mex = *3
x . x x
x x . x
x x x .
x x x x
```

For Kayles/Nim-like games (reducing number of objects in a pile or splitting piles;
can't move on more than one pile at a time [easily decomposed into subgames])

game-over position = *0
for each other initial pile P in order of increasing size
    start with S = empty set
    for each move
        determine resulting position P', write as p1 + p2 + ... + pn (objects left in each pile)
        look up what each pi is equivalent to, compute exclusive-or of all; add result to S
    compute mex(S), save that as equivalent to P

.	**K0**	**{}**	***0**
x	**K1**	**{.} = {*0}**	***1**
xx	**K2**	**{.., x., .x} = {*0, *1, *1}**	***2**
xxx	**K3**	**{.xx, x.x, ..x} = {*2, (*1 xor*1) = *0, *1}**	***3**
xxxx	**K4**	**{.xxx, x.xx, ..xx, x..x} = {*3, *(1 xor 2) = *3, *2, *(1 xor 1) = *0}**	***1**
xxxxx	**K5**	**{.xxxx, x.xxx, xx.xx, ..xxx, x..xx} = {*1, *(1 xor 3) = *2, *(2 xor 2)=*0, *3, *(1 xor 2)=*3}**	***4**

Finding winning move

0 5 10 15
0 1 2 3 1 4 3 2 1 4 2 6 4 1 2 7 1 4

Finding winning move

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4,
3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

① compute table as above up to size of largest group

xxxxx . xxxxxx . xxx . xx

*4 *3 *3 *2

② look up equivalence for each group

③ compute xor

100 move here! to 100 xor 110 = 010
011
011
010

110

④ find group that has 1 in same place as MSB of xor

possible moves on xxxxx = {.xxxx = *1, x.xxx = *(1 xor 3) =*2

⑤ compute xor of that group and result from ③

a winning move is to x.xxx.xxxxxx.xxx.xx

to disambiguate between multiple winning move in programming assignment:

```
for each group left->right
    if group has msb of xor set
        for each position left -> right
            for each move at that position fewer pins->more pins
                if move is a winning move then STOP
```

Game Positions

Game position = set of options (set of positions you can move to)

In traditional 1-row Nim

. = {}

O = {. } = { {} } = *1

O O = { ., o} = { {}, { {} } } = *2

O O O = { {} , { {} }, { {}, { {} } } } = *3

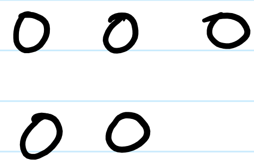
O O O O =

Outcome class: N - next player has a winning strategy -- there is a move to a P position
P - previous has winning stratgy -- all moves are to N positions

{ } = *0 is a P position

O = *1 is a N position
OO = *2 is an N position
OOO = *3 is an N position

Sums of Games



= *3 + *2

$G + H = \{ G' + H \mid G' \text{ is an option of } G \} \cup \{ G + H' \mid H' \text{ is an option of } H \}$

*3 + *2 = { *2+*2, *1+*2, *0+2, *3+*1, *3+*0}

Equivalence of Games

For impartial, normal games G, G', say $G \approx G'$ if and only if

$G + H$, $G' + H$ have same outcome class (N, P) for all impartial normal combinatorial games H

Is $*2 \approx *1$? *2 + _*1___ *1 + _*1____

No!

N				P

Is $*5 \approx *3$? *5 + _*3____ *3 + __*3______

No! N P

Conjecture: $\forall m, n \in \mathbb{N},\ m \neq n \longrightarrow$ *m is not equivalent to *n

(*m + *m is P; *n + *m is N)

Is $*2 + *1 \approx *3$ YES!

$*2 + *1 + \underline{*1}$ $*3 + \underline{*1}$

N N

$*2 + *1 + \underline{*2}$ $*3 + \underline{*2}$

N N

$*2 + *1 + \underline{*3}$ $*3 + \underline{*3}$

P P

$*2 + *1 + \underline{*4}$ $*3 + \underline{*4}$

N N

$*2 + *1 + \_\_\_$ $\quad\quad$ $*3 + \_\_\_$

Conjecture: $*n + *m \approx$ *(n xor m)