Using Sprague-Grundy

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

.	**K0**	**{}**	***0**
x	**K1**	**{.} = {*0}**	***1**
xx	**K2**	**{.., x., .x} = {*0, *1, *1}**	***2**
xxx	**K3**	**{.xx, x.x, ..x} = {*2, (*1 xor*1) = *0, *1}**	***3**
xxxx	**K4**	**{.xxx, x.xx, ..xx, x..x} = {*3, *(1 xor 2) = *3, *2, *(1 xor 1) = *0}**	***1**
xxxxx	**K5**	**{.xxxx, x.xxx, xx.xx, ..xxx, x..xx} = {*1, *(1 xor 3) = *2, *(2 xor 2)=*0, *3, *(1 xor 2)=*3}**	***4 *3**

xxxxxx K_6 {.xxxxx, x.xxxx, xx.xxx, ..xxxx, x..xxx, xx..xx}

*4   *1 + *1, *(1 ⊕ 1), *0   *2 + *3, *(2 ⊕ 3), *1   *1   *1 + *3, *(1 ⊕ 3), *2   *2 + *2, *(2 ⊕ 2), *0

mex = *3

K_5	K_6	K_{11}
xxxxx	xxxxxx	xxxxxxxxxxx
*4	*3	*6

*(4 ⊕ 3 ⊕ 6) = *1

x..xxx

$$\begin{array}{r} 100 \\ 011 \\ 110 \\ \hline 001 \end{array}$$

want to change 011 (*3) to 010 (*2)

011 ⊕ 001

Properties of Equivalence

For all finite, impartial, normal games G, H, K

$G \approx H \rightarrow$ outcome class of $G =$ outcome class of H
$G + \underline{x0}$ same outcome as $H + \underline{x0}$

$\approx$ is an equivalence relation

$G \approx G$ reflexive

$G \approx H \rightarrow H \approx G$ symmetric

$G \approx H$ and $H \approx K \rightarrow G \approx K$ transitive

$G + H \approx H + G$ commutative

$(G+H)+K \approx G+(H+K)$ associative

Lemmas

$N+N$ can be P $*5+*5$ is P $*5+*3$ is N

L1: Any position $G+H$ is an N position if G, H are in different outcome classes and is a P position if G, H are both P positions.

Proof: (Induction on length of game: $\forall n \geq 0$, $\forall$ pos $G+H$ of length n, ...)

Base case: $(n=0)$ Then $G+H = \{\ \}$ so $G = \{\ \}$ $H = \{\ \}$

Ind step: Suppose $G+H$ has length $k>0$ and suppose sums of length $<k$ satisfy

3 cases

1) G in N, H is P [want $G+H$ is an N pos]
↳ can move to P pos

Find move G' from G s.t. G' is P pos. (must be one b/c G is N)

$G'+H$ has length $<k$ so ind. hyp. applies

G' is P pos H is P pos, so $G'+H$ is P
so there's a move on $G+H$ to P pos
so $G+H$ is N pos

2) G is P, H is N
symmetric w/ 1

3) G, H both P

So every G' is N and every H' is N

Ind hyp applies to each $G'+H$ or $G+H'$: all are $N+P$ or $P+N$, so all are N

So $G+H$ is P

L2: For every P position A and every position G, $G+A \approx G$

Proof: Suppose A is a P position and G is any position

Let H be any pos.

Two cases: $G+H$ is P $G+A+H \approx \underbrace{G+H}_{P} + \underbrace{A}_{P}$
so sum $G+H+A$ is P by L1

$G+H$ is N $G+A+H \approx \underbrace{G+H}_{N} + \underbrace{A}_{P}$

N pos (L1)

L3: $G \approx G'$ if and only if $G+G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$. Then $G+G$, $G+G'$ have same outcome class
$G+G$ is P
so $G+G'$ is P too

$\leftarrow$: Suppose $G+G'$ is a P position

Then $G+(G+G') \approx G$ (L2)

and $G'+(G+G) \approx G'$ (L2)

associative
commutative

so $G \approx G+(G+G') \approx (G+G)+G' \approx G'+(G+G) \approx G'$

so $G \approx G'$ (transitive)

L4: If $G = \{G_1, \dots, G_\ell\}$ and $G_1 \approx *n_1$ and $\cdots$ and $G_\ell \approx *n_\ell$
then $G \approx \{*n_1, \dots, *n_\ell\}$

[use L3: show $G + \{*n_1, \dots, *n_\ell\}$ is P pos]

Consider options of $G + \{*n_1, \dots, *n_\ell\}$

1) $G_i + \{*n_1, \dots, *n_\ell\}$, (move on G to one of $G_1, \dots, G_\ell$)

N pos b/c has option $G_i + *n_i$ where $G_i \approx *n_i$, which is a P pos (L3)

2) $G + *n_i$, (move on $\{*n_1, \dots, *n_\ell\}$)

N pos b/c has option $G_i + *n_i$, which is a P pos

All options are N pos, so $G + \{*n_1, \dots, *n_\ell\}$ is a P pos

so $G \approx \{*n_1, \dots, *n_\ell\}$ (L3)

Every finite, impartial normal game is equivalent to some nimber.

Proof :

Base case ($n=0$): only game with length 0 is $\{\ \} = *0 \approx *0$

Induction step: Let G be a game of length $k > 0$ and suppose all games G' of length $< k$ are equivalent to some nimber.

Write $G = \{G_1, G_2, \ldots, G_\ell\}$ len of $G_i \leq$ len $G - 1$

So by induction hypothesis, can find $n_1, \ldots, n_\ell$ s.t. $G_1 \approx *n_1$ and $\ldots$ and $G_\ell \approx *n_\ell$

$$G \approx \underbrace{\{*n_1, \ldots, *n_\ell\}}_{G'} \qquad (L4)$$

Claim: $G' + *m$ is P-pos where $m = \text{mex}(\{n_1, \ldots, n_\ell\})$
so $G' \approx *m$ [will use L3: show $G' + *m$ is P pos]
$\downarrow$
show any move is to N pos

Consider all options of $G' + *m$
Three cases: i) $G' + *j$, $j < m$ (rules of $*m$)

$*j$ is an option of G'
since $j = n_i$ for some i (mex)

$*j + *j$ is P pos, so $G' + *j$ is N

ii) $*i + *m$, $i < m$

$\exists$ move $*i + *m \rightarrow \underset{P}{\underline{*i + *i}}$

So $*i + *m$ is N pos

iii) $*i + *m$, $i > m$

$\exists$ move $\underset{N}{\underline{*i + *m}} \rightarrow \underset{P}{\underline{*m + *m}}$

~~iv) $*i + *m$, $i = m$~~

All options of $G' + *m$ are N-positions
So $G' + *m$ is P

$\therefore G' \approx *m$ (L3)

$G' \approx G$

so $\sigma \approx rm$ (trans)

Theorem : $*n + *m \approx *(n \oplus m)$

Proof: (induction on length of game, $n+m$)

Base case ($n+m=0$): Then $n=0$, $m=0$, $n\oplus m=0$

$*n + *m = *0 + *0 = \{\} = *0$

Induction Step: Suppose $n+m>0$ and all n', m' s.t. $n'+m' \le n+m$ have $*n' + *m' \approx *(n'\oplus m')$

$$*n + *m = \{*0 + *m, \dots, *(n-1) + *m, \; *n + *0, \dots, *n + *(m-1)\}$$

$$\approx \{*(0\oplus m), \dots, *((n-1)\oplus m), *(n\oplus 0), \dots, *(n\oplus(m-1))\} \quad \text{(ind. hyp, L4)}$$

$$\approx *\mathrm{mex}(\{0\oplus m, \dots, (n-1)\oplus m, n\oplus 0, \dots, n\oplus(m-1)\}) \quad \text{(Sprague - Grundy)}$$

we can list options of $*n + *m$ and apply ind. hyp.

Claim: $\mathrm{mex}(\{0\oplus m, \dots, (n-1)\oplus m, n\oplus 0, \dots, n\oplus(m-1)\}) = n\oplus m$

1) $n\oplus m$ is excluded:

in 1st half → suppose $n\oplus m = i\oplus m$, $i<n$
then $n\oplus m\oplus m = i\oplus m\oplus m$
$n=i$ ⇒⇐

in 2nd half → suppose $n\oplus m = n\oplus i$, $i<m$
then $n\oplus n\oplus m = n\oplus n\oplus i$
$m=i$ ⇒⇐

2) All x s.t. $0\le x< n\oplus m$ are included:

Find most significant bit where x, $n\oplus m$ differ

1) That bit is 1 in $n\oplus m$ and 0 in x (x is smaller)

2) To be 1 in $n\oplus m$, corresponding bits in n, m are 0,1 or 1,0

Assume, WLOG, bits are 1 in n, 0 in m

3) So $m\oplus x < n$
and $*(m\oplus x) + *m$ is an option of $*n + *m$

But $*(m\oplus x) + *m \approx *(m\oplus x\oplus m)$ (ind. hyp.)
$= *x$
So $*x$ is included in

2)
n	d_1	d_2	… 1 …		
m	e_1	e_2	… 0 …		

1)
$n\oplus m$	a_1	a_2	… 1 …	b_k	
x	a_1	a_2	… 0 …	c_k	
$m\oplus x$	d_1	d_2	… 0 …		

3) lets compare n, $m\oplus x$

$e_1\oplus a_1$
$= e_1 \oplus d_1 \oplus e_1$
$= d_1$

works everywhere x, $n\oplus m$ are same, incl to left of 1st diff