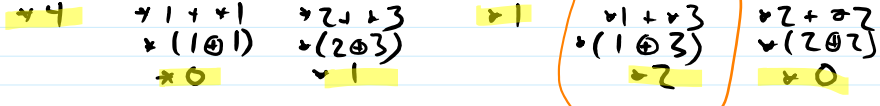


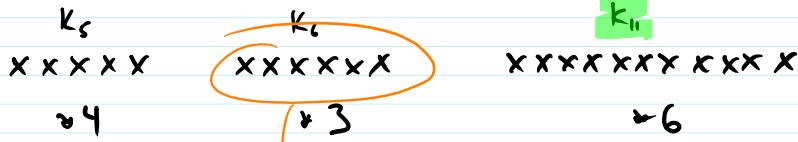
0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

.	K0	{}	*0
x	K1	{*0}	*1
xx	K2	{*, x., .x} = {*0, *1, *1}	*2
xxx	K3	{*.xx, x.x, ..x} = {*2, (*1 xor *1) = *0, *1}	*3
xxxx	K4	{.xxx, x.xx, ..xx, x..x} = {*3, *(1 xor 2) = *3, *2, *(1 xor 1) = *0}	*1
xxxxx	K5	{.xxxx, x.xxx, xx.xx, ..xxx, x..xx} = {*1, *(1 xor 3) = *2, *(2 xor 2) = *0, *3, *(1 xor 2) = *3}	*3

xxxxxx K6 { .xxxxx, x.xxxxx, xx.xxxx, ...xxxxx, x..xxxx, xx..xxx }

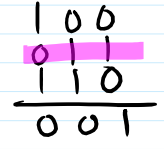


max = 3



$(4 \oplus 3 \oplus 6) = 1$

x..xxxx



want to change 011 to 010
 010 ← 011 ⊕ 001

Properties of Equivalence

For all finite, impartial, normal games G, H, K

$G \approx H \rightarrow$ outcome class of $G =$ outcome class of H
 $G + \underline{0}$ same outcome as $H + \underline{0}$

\approx is an equivalence relation

- $G \approx G$ reflexive
- $G \approx H \rightarrow H \approx G$ symmetric
- $G \approx H$ and $H \approx K \rightarrow G \approx K$ transitive
- $G + H \approx H + G$ commutative
- $(G + H) + K \approx G + (H + K)$ associative

Lemmas

$N+N$ can be P $\rightarrow 5+3$ is N
 $*5+5$ is P

L1: Any position $G+H$ is an N position if G, H are in different outcome classes and is a P position if G, H are both P positions.

Proof: (Induction on length of game: $\forall n \geq 0, \forall \text{ pos } G+H \text{ of length } n, \dots$)

Base case: ($n=0$) Then $G+H = \{\}$ so $G = \{\}$ $H = \{\}$

Ind step: Suppose $G+H$ has length $k > 0$ and suppose sums of length $< k$ satisfy

3 cases 1) G in N, H is P [want $G+H$ is an N pos]

Find move G' from G s.t. G' is P pos. (must be one b/c G is N)
 \rightarrow can move to P pos

$G'+H$ has length $< k$ so ind. hyp. applies

G' is P pos H is P pos, so $G'+H$ is P
 so there's a move on $G+H$ to P pos
 so $G+H$ is N pos

2) G is P, H is N
 symmetric w/ 1

3) G, H both P

So every G' is N and every H' is N

Ind hyp applies to each $G'+H$ or $G+H'$: all are N+P or P+N, so all are N

So $G+H$ is P

L2: For every P position A and every position G , $G+A \cong G$

Proof: Suppose A is a P position and G is any position

Let H be any pos.

Two cases: $G+H$ is P

$G+A+H \cong \underbrace{G+H}_P + \underbrace{A}_P$
 so sum $G+H+A$ is P by L1

$G+H$ is N

$G+A+H \cong \underbrace{G+H}_N + \underbrace{A}_P$
 N pos (L1)

L3: $G \approx G'$ if and only if $G + G'$ is a P position

Proof: \rightarrow : Suppose $G \approx G'$. Then $G + G$, $G + G'$ have same outcome class
 $G + G$ is P
 so $G + G'$ is P too

\leftarrow : Suppose $G + G'$ is a P position

Then $G + (G + G') \approx G$ (L2)

and $G' + (G + G) \approx G'$ (L2)

so $G \approx G + (G + G') \approx (G + G) + G' \approx G' + (G + G) \approx G'$
associative commutative

so $G \approx G'$ (transitive)

L4: If $G = \{G_1, \dots, G_r\}$ and $G_1 \approx v n_1$ and \dots and $G_r \approx v n_r$
 then $G \approx \{v n_1, \dots, v n_r\}$

[via L3]: show $G + \{v n_1, \dots, v n_r\}$ is P pos

Consider options of $G + \{v n_1, \dots, v n_r\}$

1) $G_i + \{v n_1, \dots, v n_r\}$ (move on G to one of G_1, \dots, G_r)
 \downarrow
 N pos b/c has option $G_i + v n_i$ where $G_i \approx v n_i$, which is a P pos (L3)

2) $G + v n_i$ (move on $\{v n_1, \dots, v n_r\}$)
 \downarrow
 N pos b/c has option $G_i + v n_i$, which is a P pos

All options are N pos, so $G + \{v n_1, \dots, v n_r\}$ is a P pos

so $G \approx \{v n_1, \dots, v n_r\}$ (L3)

Every finite, impartial normal game is equivalent to some number.

Proof:

Base case ($n=0$): only game with length 0 is $\{\}$ $\equiv \star 0 \cong \star 0$

Induction step: Let G be a game of length $k > 0$ and suppose all games G' of length $< k$ are equivalent to some number.

Write $G = \{G_1, G_2, \dots, G_r\}$ len of $G_i \leq \text{len } G - 1$

So by induction hypothesis, can find n_1, \dots, n_r s.t.
 $G_1 \cong \star n_1$ and \dots and $G_r \cong \star n_r$

$$G \cong \underbrace{\{\star n_1, \dots, \star n_r\}}_{G'} \quad (L4)$$

Claim: $G' + \star m$ is P-pos where $m = \max(\{n_1, \dots, n_r\})$
 so $G' \cong \star m$ [will use L3: show $G' + \star m$ is P pos]
 ↓
 show any max is to N pos

Consider all options of $G' + \star m$
 Three cases: i) $G' + \star j$, $j < m$ (rules of $\star m$)

$\star j$ is an option of G'
 since $j = n_i$ for some i (max)

$\star j + \star j$ is P pos, so $G' + \star j$ is N

ii) $\star i + \star m$, $i < m$

$$\exists \text{ max } \star i + \star m \rightarrow \underbrace{\star i + \star i}_P$$

so $\star i + \star m$ is N pos

iii) $\star i + \star m$, $i > m$

$$\exists \text{ max } \star i + \star m \rightarrow \underbrace{\star m + \star m}_P$$

~~iv) $\star i + \star m$, $i = m$~~

All options of $G' + \star m$ are N-positions
 so $G' + \star m$ is P

$$\therefore G' \cong \star m \quad (L3)$$

$$G' \cong G$$

so $G \cong \text{rm}(\text{trans})$

Theorem : $\forall n + \forall m \approx \forall (n \oplus m)$

Proof: (induction on length of game, $n+m$)

Base case ($n+m=0$): Then $n=0, m=0, n \oplus m = 0$
 $\forall n + \forall m = \forall 0 + \forall 0 = \{\} = \forall 0$

Induction Step: Suppose $n+m > 0$ and all n', m' s.t. $n'+m' \leq n+m$ have $\forall n' + \forall m' \approx \forall (n' \oplus m')$

$$\forall n + \forall m = \left\{ \forall 0 + \forall m, \dots, \forall (n-1) + \forall m, \forall n + \forall 0, \dots, \forall n + \forall (m-1) \right\}$$

$$\approx \left\{ \forall (0 \oplus m), \dots, \forall ((n-1) \oplus m), \forall (n \oplus 0), \dots, \forall (n \oplus (m-1)) \right\}$$

(ind. hyp, L4)

$$\approx \forall \text{mex}(\{0 \oplus m, \dots, (n-1) \oplus m, n \oplus 0, \dots, n \oplus (m-1)\}) \quad (\text{Sprague-Grundy})$$

we can list options of $\forall n + \forall m$ and apply ind. hyp.

Claim: $\text{mex}(\{0 \oplus m, \dots, (n-1) \oplus m, n \oplus 0, \dots, n \oplus (m-1)\}) = n \oplus m$

1) $n \oplus m$ is excluded: suppose $n \oplus m = i \oplus m, i < n$
 then $n \oplus m \oplus m = i \oplus m \oplus m$
 $n = i \Rightarrow \Leftarrow$
 in 1st half \rightarrow

in 2nd half \rightarrow suppose $n \oplus m = n \oplus i, i < m$
 then $n \oplus n \oplus m = n \oplus n \oplus i$
 $m = i \Rightarrow \Leftarrow$

2) All x s.t. $0 \leq x < n \oplus m$ are included:

Find most significant bit where $x, n \oplus m$ differ

1) That bit is 1 in $n \oplus m$ and 0 in x (x is smaller)

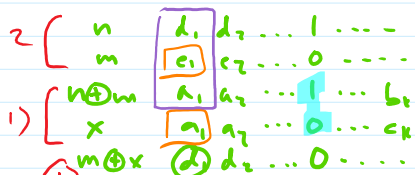
To be 1 in $n \oplus m$, corresponding bits in n, m are 0,1 or 1,0

2)

Assume, wlog, bits are 1 in $n, 0$ in m

So $m \oplus x < n$ and $\forall (m \oplus x) + \forall m$ is an option of $\forall n + \forall m$

But $\forall (m \oplus x) + \forall m \approx \forall (m \oplus x \oplus m)$ (ind. hyp.)
 $= \forall x$
 so x is included in



3) let's compare $n, m \oplus x$

$$\begin{aligned} & e_1 \oplus a_1 \\ & = e_1 \oplus d_1 \oplus e_1 \\ & = d_1 \end{aligned}$$

works everywhere
 $x, n \oplus m$ are same, incl to left of 1st diff