Simultaneous Play Games

	R	P	S
Rock	0, 0	−1, 1	1, −1
Paper	1, −1	0, 0	−1, 1
Scissors	−1, 1	1, −1	0, 0

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

a_{ij} = payoff for PI when
PI chooses row i
PII chooses col j

$$B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

b_{ij} = payoff for PII

Zero-sum: $a_{ij} + b_{ij} = 0$ for all i,j

constant-sum: $a_{ij} + b_{ij} = c$ for all i,j

Penalty Kick

goalkeeper

kicker		L	R
	L	$\frac{1}{8}, -\frac{1}{8}$	$\frac{1}{4}, -\frac{1}{4}$
	R	$\frac{1}{4}, -\frac{1}{4}$	$\frac{1}{6}, -\frac{1}{6}$

expected change in win pct (so zero-sum)

PI's payoffs

$$A \quad \begin{array}{c|cccc} & W & X & Y & Z \\ \hline P & & & & \\ Q & \multicolumn{4}{c}{\begin{pmatrix} \frac{2}{3} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ -1 & \frac{1}{3} & \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -1 & \frac{1}{2} \end{pmatrix}} \\ R & & & & \end{array}$$

if zero-sum then knowing A determines B

$$B = -A = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & -\frac{3}{4} & -\frac{3}{4} \\ 1 & -\frac{1}{3} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

		R	P	S
	R	$\frac{1}{2}, \frac{1}{2}$	0, 1	1, 0
I	P	1, 0	$\frac{1}{2}, \frac{1}{2}$	0, 1
	S	0, 1	1, 0	$\frac{1}{2}, \frac{1}{2}$

Rock, Paper, Scissors as constant-sum game
(payoff = # wins ; draw = $\frac{1}{2}$ win)

Stag Hunt

	S	H
Stag	2, 2	0, 1
Hare	1, 0	1, 1

non-constant sum

Prisoner's Dilemma

	C	D
Cooperate	−1, −1	−10, 0
Defect	0, −10	−5, −5

constant-sum game w/ constant $C \equiv$ zero-sum game

1) play zero-sum game on $A - \frac{c}{2}$, $B - \frac{c}{2}$

no choices here → 2) award PI $\frac{c}{2}$, PII $\frac{c}{2}$

→ optimizing 1+2 = optimizing 1

equilibrium: neither player has incentive to change unilaterally

	R	P	S	
Rock	0	−1	1	−1
Paper	1	0	−1	−1
Scissors	−1	1	0	−1
	1	1	1	$v^+ = 1$

$v^- = -1$

	L	R	min
L	$\frac{1}{2}$	1	$\frac{1}{2}$
R	1	$\frac{2}{3}$	$\frac{2}{3}$
max	1	1	

$v^- = \frac{2}{3}$, $v^+ = 1$

	W	X	Y	Z	
P	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2} = v^-$
Q	−1	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$	−1
R	$\frac{1}{2}$	0	−1	$\frac{1}{2}$	−1
	$\frac{2}{3}$	$\frac{1}{2} = v^+$	$\frac{3}{4}$	$\frac{3}{4}$	

v^- : amount I is guaranteed to win
$= \max_i \min_j a_{ij}$

v^+ : ceiling on what II gives up
$= \min_j \max_i a_{ij}$

For any constant-sum game, $v^- \leq v^+$

$\forall i,j \quad \min_{j'} a_{ij'} \leq a_{ij}$ — any entry is ≥ min of entries in same row (def min)

$\forall j \quad \max_i \min_{j'} a_{ij'} \leq \max_i a_{ij}$ — max of smaller things ≤ max of larger things

$x_1 < y_1$
$x_2 < y_2$
$\vdots$
$x_n < y_n$

$\min_j \max_i \min_{j'} a_{ij'} \leq \min_j \max_i a_{ij}$ — min of smaller things ≤ min of larger things

$\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}$ — j does not appear in terms on left

$v^- \leq v^+$

Nash equilibrium: where neither player has incentive to unilaterally change strategy

pick one row
one column

Nash equilibrium: where neither player has incentive to unilaterally change strategy

pick one row
one column

A constant-sum game A has an equilibrium in pure strategies if and only if $v^- = v^+$

$\Rightarrow$: Suppose A has a saddle point in pure strategies.

Then $\exists\ i^*, j^*$ s.t. $\forall\, i,j \quad a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$

so $\max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j}$ max of terms all $\le a_{i^*j^*}$ still $\le a_{i^*j^*}$ (similar for min)

and $\underbrace{\min_j \max_i a_{ij}}_{v^+} \le \max_i a_{ij^*}$ $\quad \min_j a_{i^*j} \le \underbrace{\max_i \min_j a_{ij}}_{v^-}$ def min/max

so $v^+ \le \max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j} \le v^-$ so $v^+ \le v^-$

but $\underbrace{v^- \le v^+}_{\text{prev thm}}$ so all $\le$ are $=$ and so $v^- = v^+$

$\Leftarrow$ Suppose $v^- = v^+$. Let i^* be the i s.t. $v^- = \max_i \min_j a_{ij} = \min_j a_{i^*j}$

j^* be the j s.t. $v^+ = \min_j \max_i a_{ij} = \max_i a_{ij^*}$

def min $\quad a_{i^*j} \ge \min_j a_{i^*j} = v^+ = v^- = \max_i a_{ij^*} \ge a_{ij^*}$ def max for all i,j

$a_{i^*j^*} \ge v^+ = v^- \ge a_{i^*j^*}$ use $i = i^*$, $j = j^*$,

and $\ge$ are $=$

$\therefore\ v^+ = v^- = a_{i^*j^*}$

$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$ for all i,j sub $a_{i^*j^*}$ in

Suppose there are 2 equilibria in pure strategies (i_1, j_1) and (i_2, j_2)

with values $a_{i_1j_1} = v_1 \quad a_{i_2j_2} = v_2$

$$\begin{pmatrix} x & b \\ a & y \end{pmatrix}$$

x and y are both equilibria

$x \le b \le y \le a \le x$

but $x = x$ (not $x < x$) so all $\le$ are $=$

$x = b = y = a = x$

(and a, b are equilibria too)

Mixed Strategies - probability distribution $(x_1, \ldots, x_n)$ $x_i = P(\text{PI chooses row } i)$

$(y_1, \ldots, y_m)$ $y_j = P(\text{PII chooses col } j)$

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$X = (\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}) = Y$

$$E[X,Y] = \sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I chooses row } i \text{ and II chooses col } j)\cdot a_{ij}$$ expected value formula

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I chooses row } i)\cdot P(\text{II chooses col } j)\cdot a_{ij}$$ independent

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \cdot y_j \cdot a_{ij} = XAY^T$$

$$= \tfrac{1}{3}\cdot\tfrac{1}{3}\cdot 0 + \tfrac{1}{3}\cdot\tfrac{1}{3}\cdot 1 + \tfrac{1}{3}\cdot\tfrac{1}{3}\cdot -1 + \cdots$$

$$= 0$$

X^*, Y^* is equilibrium means

$$E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$$ for all mixed strategies X, Y

Every game has an equilibrium in mixed strategies

Finding Equilibrium in Mixed Strategies

Thm: X^*, Y^* is an equilibrium in mixed strategies and value$(A) = E(X^*, Y^*)$

if and only if

$$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j) \quad \text{for all } i, j$$

pure strat with $x_i = 1$, $x_k = 0$ for $l \neq l$

$y_j = 1$, $y_k = 0$ for $k \neq j$

Proof: next class

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Claim $X^* = \left(\frac{2}{7}, 0, \frac{5}{7}\right)$ $\quad Y^* = \left(\frac{5}{7}, \frac{2}{7}, 0\right)$

$$E(X^*, Y^*) = \frac{2}{7}\cdot\frac{5}{7}\cdot 0.30 + \frac{2}{7}\cdot\frac{2}{7}\cdot 0.25 + \frac{5}{7}\cdot\frac{5}{7}\cdot 0.28 + \frac{5}{7}\cdot\frac{2}{7}\cdot 0.30$$
$$= \frac{2}{7}$$

$E(X^*, 1) = \frac{2}{7}\cdot 0.30 + \frac{5}{7}\cdot 0.28 = \frac{2}{7} \ge \frac{2}{7}$ ✓

$E(X^*, 2) = \frac{2}{7} \ge \frac{2}{7}$ ✓

$E(X^*, 3) = \frac{205}{700} \ge \frac{2}{7}$ ✓

$E(1, Y^*) \le E(X^*, Y^*)$ ✓

$\frac{5}{7}\cdot 0.30 + \frac{2}{7} 0.25 = \frac{2}{7} \le \frac{2}{7}$ ✓

$E(2, Y^*) = \frac{5}{7}\cdot 0.26 + \frac{2}{7}\cdot 0.33 = \frac{196}{700} \le \frac{2}{7}$ ✓

$E(3, Y^*) = \frac{2}{7} \le \frac{2}{7}$ ✓

Suppose there are two equilibria in mixed strategies (X_1^*, Y_1^*) (X_2^*, Y_2^*)

$$\begin{aligned}
E(X_1^*, Y_2^*) &\le E(X_2^*, Y_2^*) && \text{def equilibrium} \\
&\le E(X_2^*, Y_1^*) && \text{"} \\
&\le E(X_1^*, Y_1^*) && \text{"} \\
&\le E(X_1^*, Y_2^*) && \text{"}
\end{aligned}$$

so all $\le$ are $=$ (and all are equilibria)