Finding Equilibrium in Mixed Strategies

Thm: X^*, Y^* is an equilibrium in mixed strategies and $\text{value}(A) = E(X^*, Y^*)$

if and only if

$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$ for all i, j

pure strategy – pick row i = mixed strategy with $x_i = 1$, $x_k = 0$ for $k \ne j$ $(0 \dots 0\, 1\, 0 \dots 0)$, x_i

$y_j = 1$, $y_k = 0$ for $k \ne j$ $(0 \dots 0\, 1\, 0 \dots 0)$, y_j

$\Rightarrow$ Suppose X^*, Y^* is an equilibrium then $\forall X, Y$ $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$ def eq.

$\forall i, j$ $E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$ special cases

$\Leftarrow$ Suppose $E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$ for all i, j

Let $X = (x_1, \dots, x_n)$ where $0 \le x_i \le 1$ and $\Sigma x_i = 1$

$$\begin{aligned} E(X, Y^*) = \; & x_1 \cdot E(1, Y^*) & \le \; & x_1 \cdot E(X^*, Y^*) \\ + \; & x_2 \cdot E(2, Y^*) & & x_2 \cdot E(X^*, Y^*) \\ & \vdots & & \vdots \\ + \; & x_n \cdot E(n, Y^*) & & x_n \cdot E(X^*, Y^*) \end{aligned}$$

$(x_1 + \dots + x_n) E(X^*, Y^*) = E(X^*, Y^*)$

$\forall X, Y$ $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$

Let $Y = (y_1, \dots, y_m)$

$$\begin{aligned} y_1 \cdot E(X^*, Y^*) &\le y_1 \cdot E(X^*, 1) \\ &\vdots \\ y_m \cdot E(X^*, Y^*) &\le y_m \cdot E(X^*, 1) \end{aligned}$$

$E(X^*, Y^*) \le E(X^*, Y)$

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Claim $X^* = (\frac{2}{7}, 0, \frac{5}{7})$ $Y^* = (\frac{5}{7}, \frac{2}{7}, 0)$

$E(X^*, Y^*) = \frac{2}{7} \cdot \frac{5}{7} \cdot 0.30 + \frac{2}{7} \cdot \frac{2}{7} \cdot 0.25 + \frac{5}{7} \cdot \frac{5}{7} \cdot 0.28 + \frac{5}{7} \cdot \frac{2}{7} \cdot 0.30$

Verify $X^* = (\frac{2}{7}\ 0\ \frac{5}{7})$ $Y^* = (\frac{5}{7}\ \frac{2}{7}\ 0)$ is equilibrium $= \frac{2}{7}$

$v = E(X^*, Y^*) = \frac{2}{7}$

$E(1, Y^*) = \frac{5}{7} \cdot 0.30 + \frac{2}{7} \cdot 0.25 = \frac{2}{7} \le \frac{2}{7}$

$E(2, Y^*) = \dots = \frac{196}{700} \le \frac{2}{7}$

$E(3, Y^*) = \dots = \frac{2}{7} \le \frac{2}{7}$

$E(X^*, 1) = \frac{2}{7} \cdot 0.30 + \frac{5}{7} \cdot 0.28 = \frac{200}{700} = \frac{2}{7} \ge \frac{2}{7}$

$E(X^*, 2) = \dots = \frac{200}{700} = \frac{2}{7} \ge \frac{2}{7}$

$E(X^*, 3) = \dots = \frac{205}{700} \ge \frac{2}{7}$

Is $X = (\frac{1}{2}\ \frac{1}{2}\ 0)$ $Y = (\frac{1}{2}\ \frac{1}{2}\ 0)$ an equilibrium

$E(X, Y) = 0.285$

$E(1, Y) = 0.275$

$E(2, Y) = 0.295$ (not $\le$) $> .285 \Rightarrow$ not equilibrium

$E(3, Y) = 0.29$

$E(X, 1) = \frac{1}{2} \cdot 0.30 + \frac{1}{2} \cdot 0.26 = 0.27$ not ≥ 0.285 not equilibrium

$E(X, 2) = 0.29$

$E(X, 3) = 0.24$

↳ best response

	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$

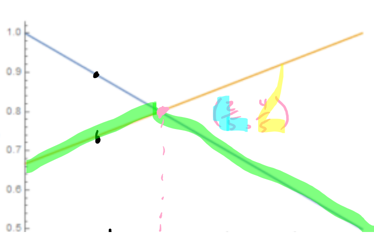
$X = (x \quad 1-x)$

$E(X, 1) = x \cdot \frac{1}{2} + (1-x) \cdot 1$
$= 1 - \frac{1}{2} x$

$E(X^*, Y^*) \le E(X^*, 1)$

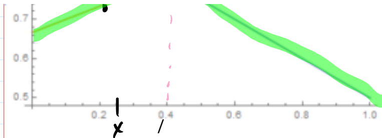
$E(X, 2) = x \cdot 1 + (1-x) \frac{2}{3}$
$= \frac{2}{3} + \frac{1}{3} x$

$E(X^*, Y) \le E(X^*, 2)$



best response to X is $\min(E(X,1), E(X,2))$

equilibrium is at max of best response = intersection of $E(X,1)$, $E(X,2)$



equilibrium is at max of best response
= intersection of $E(x,1), E(x,2)$

$1-\frac{1}{2}x = \frac{2}{3}+\frac{1}{3}x$

$\frac{1}{3} = \frac{5}{6}x$

$x = \frac{2}{5}$

$x^* = \left(\frac{2}{5}\ \ \frac{3}{5}\right)$

$v = 1-\frac{1}{2}\cdot\frac{2}{5}$
$= \frac{2}{3}+\frac{1}{3}\cdot\frac{2}{5} = \frac{4}{5}$

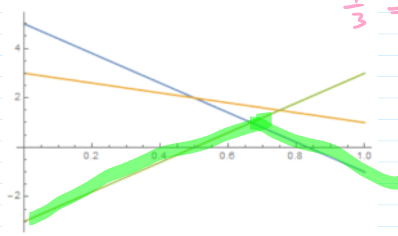
$E(1,y) = \frac{1}{2}y + 1\cdot(1-y) = 1-\frac{1}{2}y$

$E(2,y) = y+\frac{2}{3}(1-y) = \frac{2}{3}+\frac{1}{3}y$

(same graph b/c symmetry)

$y^* = \left(\frac{2}{5}\ \ \frac{3}{5}\right)$

$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$



$E(x,1) = x+3(1-x) = 3-2x$

$E(x,2) = -1x+5(1-x) = 5-6x$

$E(x,3) = 3x+{-3}(1-x) = -3+6x$

col 1 not used in equilibrium
(not on upper envelope around intersection)
so $y^* = (0\ \ y\ \ 1-y)$

$E(1,y) = -1y+3(1-y) = 3-4y$

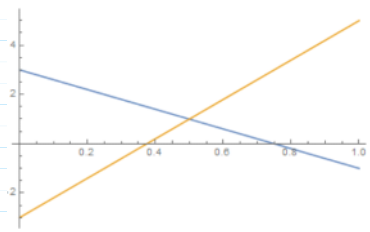
$E(2,y) = 5y+{-3}(1-y) = -3+8y$

$3-4y = -3+8y$

$6 = 12y$

$\frac{1}{2} = y$

$y^* = \left(0\ \ \frac{1}{2}\ \ \frac{1}{2}\right)$



$A = \begin{pmatrix} 0 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$

Linear Programming

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Maximize v = minimize $\frac{1}{v}$
subject to

$E(X,1) = 0.30x_1 + 0.26x_2 + 0.28x_3 \geq v$

$E(X,2) = 0.25x_1 + 0.33x_2 + 0.30x_3 \geq v$

$E(X,3) = 0.20x_1 + 0.28x_2 + 0.33x_3 \geq v$

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

$\frac{x_1}{v} = p_1 \quad \frac{x_2}{v} = p_2 \ldots$

divide by v and $\times -1$

$0.20 \leq v \leq 0.33$

min payoff max payoff

positive, so :) (if not, add C to all payoffs to make positive) and subtract out after getting soln for value of original game

a_1 b_ub

$-0.30p_1 - 0.26p_2 - 0.28p_3 \leq -1$

$-0.25p_1 - 0.33p_2 - 0.30p_3 \leq -1$

$-0.20p_1 - 0.28p_2 - 0.33p_3 \leq -1$

for Java, convert to =

$-0.30p_1 - 0.26p_2 - 0.28p_3 + 1s_1 + 0s_2 + 0s_3 = -1$

$-0.25p_1 - 0.33p_2 - 0.33p_3 + 0s_1 + 1s_2 + 0s_3 = -1$

$\vdots$

$0 \leq s_i \leq 1$

$\vdots$

bounds

$0 \leq p_1 \leq 5$

$0 \leq p_2 \leq 5$

$0 \leq p_3 \leq 5$

$p_i = \frac{x_i}{v}$

$x_i = 1$

$v \geq .2$

$\frac{1}{v} \leq 5$

minimize $p_1 + p_2 + p_3 = \frac{x_1}{v} + \frac{x_2}{v} + \frac{x_3}{v} = \frac{x_1+x_2+x_3}{v} = \frac{1}{v}$

c

lin prog returns p_i's and $\frac{1}{v}$; convert back to x_i's and v

II

minimize v subject to

$E(1,Y) = 0.30y_1 + 0.25y_2 + 0.20y_3 \leq v$

$E(2,Y) = 0.26y_1 + 0.33y_2 + 0.28 \leq v$

$E(3,Y) = 0.28y_1 + 0.30y_2 + 0.33 \leq v$

$y_1 + y_2 + y_3 = 1$

$y_1, y_2, y_3 \geq 0$

$\Downarrow$ divide by v, set $q_i = \frac{y_i}{v}$
(don't need to multiply by -1 b/c already $\leq$)

Easier transform leaving v
takes longer
418017 vs 197123 iter
through t=17 in

a_2

$0.30q_1 + 0.25q_2 + 0.20q_3 \leq 1$

takes longer

5:20 vs 3:00 tot time (incl matrix setup)

418017 vs 197123 iter through t=12 in two-minute init and fails sometimes

$\left(\sum x_i = \sum y_i = 1 \right.$ too constraining?$\left.\right)$

no crash w/no bounds on v and down to 384063 iter

1657815 in 15:41

869563 in 10:05

(don't know how much time is linprog vs other stuff)

$$\underbrace{\begin{matrix} 0.30\,g_1 + 0.25\,g_2 + 0.20\,g_3 \\ 0.26\,g_1 + 0.33\,g_2 + 0.28\,g_3 \\ 0.28\,g_1 + 0.30\,g_2 + 0.33\,g_3 \end{matrix}}_{a_2} \begin{matrix} \le \\ \le \\ \le \end{matrix} \underbrace{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}}_{\text{bub}}$$

$$\begin{matrix} 0 \le g_1 \le 5 \\ 0 \le g_2 \le 5 \\ 0 \le g_3 \le 5 \end{matrix} \quad \text{bounds}$$

minimize $\underbrace{-g_1 + -g_2 + -g_3}_{c} = -\frac{1}{v}$ (same as max $\frac{1}{v}$, same as min v)

non-constant sum

$$E_I(X,Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \underline{x_i \cdot y_j} \cdot a_{ij}$$

nonlinear (and appear in objective fxn in system to optimize)

$$E_{II}(X,Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \underline{x_i \cdot y_j} \cdot b_{ij}$$

non-linear programming