## Game Positions

Game position =

In traditional 1-row Nim

=

O =

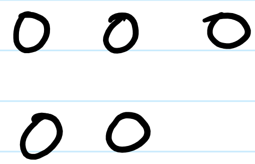
O O =

O O O =

O O O O =

$\{\,\} = *0$ is a position

## Sums of Games



$=$

$G + H = \{ \qquad \}$

$\cup$

$\{ \qquad \}$

## Equivalence of Games

For impartial, normal games $G, G'$, say $G \approx G'$ if and only if

$G + H, \quad G' + H$

Is $*2 \approx *1$ ?

Is $*5 \approx *3$ ?

Conjecture: $\forall m, n \in \mathbb{N}, \; m \neq n \longrightarrow$

Is $*2 + *1 \approx *3$

$*2 + *1 + \_\_\_ \qquad *3 + \_\_\_$

$*2 + *1 + \_\_\_ \qquad *3 + \_\_\_$

$*2 + *1 + \_\_\_ \qquad *3 + \_\_\_$

$*2 + *1 + \_\_\_ \qquad *3 + \_\_\_$

$*2 + *1 + \_\_\_$ $*3 + \_\_\_$

Conjecture: $*n + *m \approx$

**Properties of Equivalence**

For all finite, impartial, normal games $G, H, K$

$$G \approx H \rightarrow$$

$$G \approx$$

$$G \approx H$$

$$G \approx H \text{ and } H \approx K$$

$$G + H \approx$$

$$(G + H) + K \approx$$

Lemmas

L1: Any position $G+H$ is an $N$ position if $G, H$ are in different outcome classes and is a $P$ position if $G, H$ are both $P$ positions.

Proof: (Induction on length of game: $\forall n \geq 0$, $\forall$ pos $G+H$ of length $n$, ...)

Base case: $(n=0)$ Then $G+H = \{\}$ so $G=$ $H=$

Ind step: Suppose $G+H$ has length $k>0$ and suppose sums of length $<k$ satisfy

3 cases 1) $G$ in $N$, $H$ is $P$

2) $G$ is $P$, $H$ is $N$

3) $G, H$ both $P$

So every $G'$ is $N$ and every $H'$ is $N$

Ind hyp applies to each $G'+H$ or $G+H'$: all are $N+P$ or $P+N$, so all are $N$

So $G+H$ is $P$

L2: For every $P$ position $A$ and every position $G$, $G+A \approx G$

Proof: Suppose A is a P position and $G$ is any position

Let $H$ be any pos.

Two cases: $G+H$ is $P$ $\qquad G+A+H \approx G+H+A$

$G+H$ is $N$ $\qquad G+A+H \approx \underbrace{G+H}_{N}+\underbrace{A}_{P}$

$N$ pos (L1)

L3: $G \approx G'$ if and only if $G+G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$. Then $G+G$, $G+G'$ have same outcome class
$G+G$ is P
so $G+G'$ is P too

$\leftarrow$: Suppose $G+G'$ is a P position

Then $G+(G+G') \approx G$ (L2)

and $G'+(G+G) \approx G'$ (L2)

associative
commutative

so $G \approx G+(G+G') \approx (G+G)+G' \approx G'+(G+G) \approx G'$

so $G \approx G'$ (transitive)

L4: If $G=\{G_1, \ldots, G_\ell\}$ and $G_1 \approx *n_1$ and $\cdots$ and $G_\ell \approx *n_\ell$
then $G \approx \{*n_1, \ldots, *n_\ell\}$

[use L3: show $G+\{*n_1, \ldots, *n_\ell\}$ is P pos]

Consider options of $G+\{*n_1, \ldots, *n_\ell\}$

1) $\underbrace{G_i + \{*n_1, \ldots, *n_\ell\}}$, (move on $G$ to one of $G_1, \ldots, G_\ell$)

N pos b/c has option $G_i + *n_i$ where $G_i \approx *n_i$, which is a P pos (L3)

2) $\underbrace{G + *n_i}$, (move on $\{*n_1, \ldots, *n_\ell\}$)

N pos b/c has option $G_i + *n_i$, which is a P pos

All options are N pos, so $G+\{*n_1, \ldots, *n_\ell\}$ is a P pos

so $G \approx \{*n_1, \ldots, *n_\ell\}$ (L3)

Every finite, impartial normal game is equivalent to some nimber.

Proof :

Base case ($n=0$): only game with length $0$ is $\{\,\}$ $= *0 \approx *0$

Induction step: Let $G$ be a game of length $k>0$ and suppose all games $G'$ of length $<k$ are equivalent to some nimber.

So by induction hypothesis,

Claim: $G' + *m$ is P-pos where $m = \text{mex}(\{n_1, \dots, n_\ell\})$
so $G' \approx *m$

Consider all options of $G' + *m$
Three cases: i) $G' + *j$,

ii) $*i + *m$

iii) $*i + *m$,

iv) $*i + *m$,

All options of $G' + *m$ are N-positions
So $G' + *m$ is P

All options of $G' + *m$ are $N$-positions
so $G' + *m$ is $P$

$\therefore G' \approx *m$ (L3)

$G' \approx G$

so $G \approx *m$ (trans)

Nim-sum

Theorem : $*n + *m \approx *(n \oplus m)$

Proof: (induction on length of game, $n+m$)

Base case ($n+m=0$): Then $n=0$, $m=0$, $n \oplus m = 0$

$*n + *m = *0 + *0 = \{\} = *0$

Induction Step: Suppose $n+m>0$ and all $n', m'$ s.t. $n'+m' \leq n+m$ have $*n' + *m' \approx *(n' \oplus m')$

$$*n + *m = \{*0 + *m, \dots, *(n-1) + *m, *n + *0, \dots, *n + *(m-1)\}$$

$$\approx \{*(0 \oplus m), \dots, *((n-1) \oplus m), *(n \oplus 0), \dots, *(n \oplus (m-1))\}$$

$$\approx *\mathrm{mex}(\{0 \oplus m, \dots, (n-1) \oplus m, n \oplus 0, \dots, n \oplus (m-1)\})$$

Claim: $\mathrm{mex}(\{0 \oplus m, \dots, (n-1) \oplus m, n \oplus 0, \dots, n \oplus (m-1)\}) = n \oplus m$

1) $n \oplus m$ is excluded: suppose $n \oplus m = i \oplus m$, $i < n$

then $n \oplus m \oplus m = i \oplus m \oplus m$

$n = i$

suppose $n \oplus m = n \oplus i$, $i < m$

then $n \oplus n \oplus m = n \oplus n \oplus i$

$m = i$

2) All $x$ s.t. $0 \leq x < n \oplus m$ are included:

Find most significant bit where $x$, $n \oplus m$ differ

That bit is in $n \oplus m$ and in $x$

To be in $n \oplus m$, corresponding bits in $n, m$ are

Assume, WLOG, bits are in $n$, in $m$

So $m \oplus x < n$

and $*(m \oplus x) + *m$ is an option of $*n + *m$

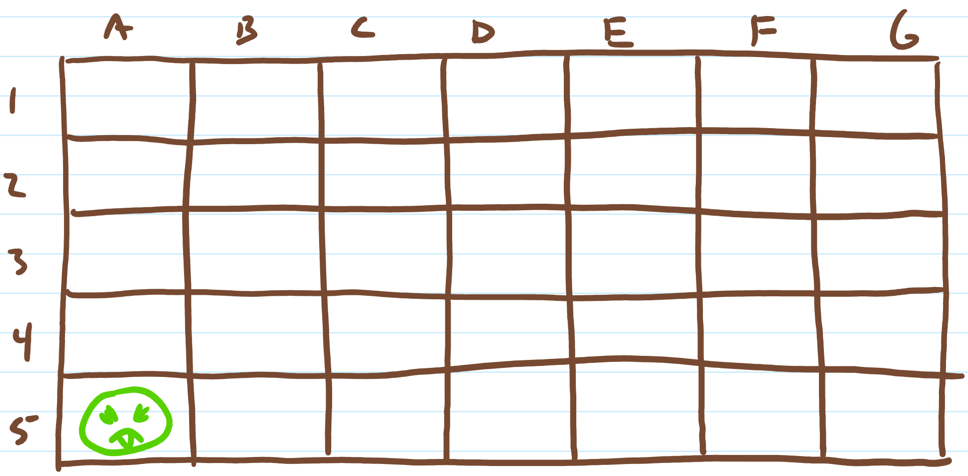
But $*(m \oplus x) + *m \approx *(m \oplus x \oplus m)$

$= *x$

## Chomp

Play on m × n grid. Take turns selecting a remaining cell, remove all above and to right.

Last move <u>loses</u>




Outcome class = who has winning strategy

N    next player wins

P    prev player wins

Any position in a finite, impartial, normal or misere game is

0 0 0
1 0 0
1 1 0
1 1 1
2 0 0
2 1 0
2 1 1
2 2 0
2 2 1
2 2 2