CPSC 474/574 - Fall 2018 - Exam #1

Write your name and NetID on only this page of this exam package in the boxes provided and write your answers on the front of the provided sheets no closer than $\frac{1}{2}$ inch from the edges of the pages.

Name

NetID

Problem 1 (10 points): Consider two finite, impartial combinatorial games with the same movement rules (for any position, the same moves are possible in both games) but where one is misere (last move loses) and one is normal (last move wins). Is it always the case that the same position will be in different outcome classes in the two games (that is, an N position in one and a P position in the other)? Explain your answer.

No : 1-2-3 Nim

misere : N P N N N P N N N P . . .

normal : P N N N . . .

Same outcome class!

Problem 2 (10 points): Find a winning move for the position shown below in this variant of Kayles if such a move exists; otherwise simply state that no such move exists. (Grundy numbers for some initial positions of the variant are given below for reference.)

~~xxxxxx xxxx x~~

Kayles Position		x	xx	xxx	xxxx	xxxxx	xxxxxx
Grundy Number	0	1	2	3	4	1	6

XXX~~XX~~X XXXXX X XX

*6 *1 *1 *2

110
001
001
010
———
100

110
100
———
010

change 6 to 2

x x x ~~x x~~ x

*3 *1

3⊕1=2

Problem 3 (16 points): Consider the finite, impartial, normal combinatorial game that is played with piles of stones. On each turn, moves available are

1. remove a single stone from a non-empty pile; or
2. split a pile containing at least 2 stones into smaller piles, each with the same number of stones (so a pile of 2 can be reduced to a pile of 1 or split into two piles of 1; a pile of 6 can be reduced to a pile of 5 or split into six piles of 1, three piles of 2, or two piles of 3).

(a) Suppose game G starts with two piles containing n_1 and n_2 stones respectively. Is this game equal, for all n_1 and n_2, to $H_1 + H_2$ where H_1 is the game that starts with a single pile of n_1 stones and H_2 is the game that starts with a single pile of n_2 stones? Explain your answer.

Yes: move on H_1+H_2 is by def a move on H_1 or on H_2, which are available in G as moves on 1^{st} or 2^{nd} pile respectively. A move on G is a move on either 1^{st} or 2^{nd} pile (not both) so is available in H_1+H_2 as a move on H_1 or H_2 respectively.

(b) For $n = 0, \ldots, 7$, find the Grundy number of the game that starts with a single pile of n stones.

0	$*0$
1	$*1$
2	$*2$
3	$*0$
4	$*1$
5	$*0$
6	$*1$
7	$*0$

1	11		$*1$ $*0$
2	111		$*2$ $*1$
3	1111	22	$*0$ $*0$ $*0$
4	11111		$*1$ $*1$
5	111111	22 333	$*0$ $*0$ $*2$ $*0$
6	1111111		

(c) Consider the variant of that game that allows an additional kind of move: the player may select a pile and remove a single stone from that pile and also all the other piles of exactly the same size. Is $G = H_1 + H_2$ for this variant, where G, H_1, and H_2 are otherwise as described in (a)? Explain your answer.

No: G now allows moves on both piles at once but H_1+H_2 by def allows moves on only one.

Problem 4 (16 points): Consider a solitaire game played with k coins placed on spaces numbered from 0 at the left to $n-1$ on the right. Each space may be empty or contain one coin (so $k < n$). On each turn, the player selects an empty space with at least one coin somewhere to its right and, randomly and uniformly, one of the coins to the right moves to the first empty space to that coin's left. For example, if there are coins in spaces $\{1, 3, 4, 6\}$ and the player selects space 2, then with probability $\frac{1}{3}$ the coin at space 3 moves to space 2, with probability $\frac{1}{3}$ the coin at space 4 moves to space 2, and with probability $\frac{1}{3}$ the coin at space 6 moves to space 5. The game ends when the coins are in spaces $\{0, \ldots, k = 1\}$. The goal is to finish the game is as few moves as possible.

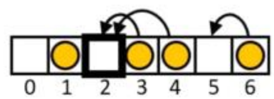


Figure 1: The three equally likely outcomes after selecting space 2

For a set S of spaces containing coins, let $E(S)$ be the expected number of turns required to complete the game when following the optimal strategy (the strategy that minimizes $E(S)$). Write a recurrence (including the base cases) for $E(S)$ and, assuming you have the values of $E(S)$, show how to compute $OPT(S)$, which is the space to choose from position S when following the optimal strategy. In addition to the usual set operations (including size, membership, union, and set difference), you may use the following functions.

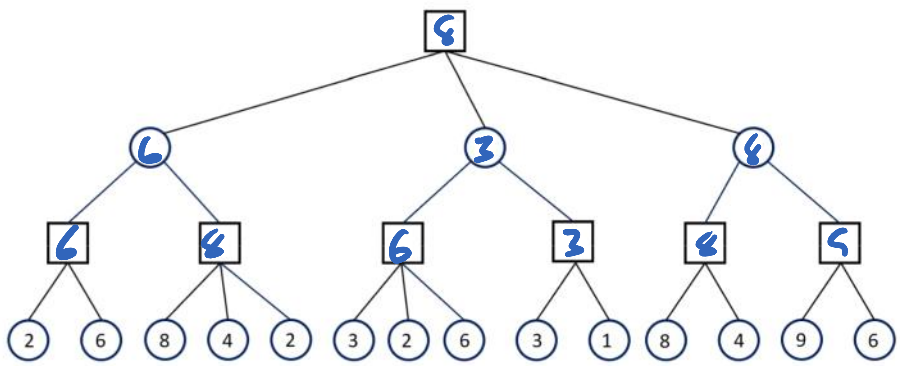
- rightmost(S): when S is the set of spaces containing coins, returns the rightmost space containing a coin (the maximum value in S: for example, rightmost($\{0, 1, 4\}$) $= 4$)
- rightof(S, i): when S is the set of spaces containing coins and i is a space so that $i \notin S$ and $i <$ rightmost(S), returns the subset of spaces in S that are to the right of space i (the elements of S greater than i: for example, rightof($\{0, 1, 4, 6\}, 2$) $= \{4, 6\}$)
- moveto(S, i): when S is the set of spaces containing coins and i is a space containing coin that has an empty space somewhere to its left, returns the position of the rightmost empty space to the left of position i (the largest integer less than i that is not in S: for example, moveto($\{0, 2, 3, 4\}, 3$) $= 1$).

$$E(\{0, \ldots, k-1\}) = 0$$

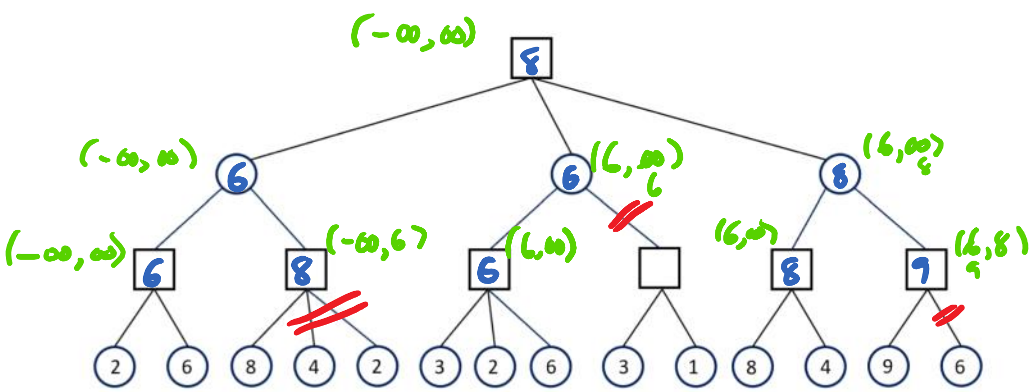
$$E(S) = 1 + \min_{\substack{c < \text{rightmost}(S) \\ c \notin S}} \underbrace{\sum_{i \in \text{rightof}(S,c)} \frac{1}{|\text{rightof}(S,c)|} \cdot E(S - \{i\} + \{\text{moveTo}(S,i)\})}$$

$$OPT(S) = \underset{\text{same bounds}}{\arg\min}$$

Problem 5 (8 points): Compute the Minimax values for the given game tree (squares are max nodes and circles are min nodes).



Problem 6 (12 points): Illustrate the operation of Alpha-Beta pruning on the given game tree by showing the (α, β) window passed to each non-leaf and which branches are pruned (you need not indicate the values returned).



Problem 7: (12 points) Find a saddle point in mixed strategies and its value for the constant-sum game with payoff matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

$X^* = (x_1 \quad 1-x_1)$ $\qquad E(X^*, 1) = 2x_1 + 1 - x_1 \geq v$

$Y^* = (y_1 \quad 1-x_1)$ $\qquad E(X^*, 2) = -x_1 + 3(1-x_1) \geq v$

intersection $\quad x_1 + 1 = -4x_1 + 3$

$5x_1 = 2$

$x_1 = \frac{2}{5} \qquad x^* = \left(\frac{2}{5} \; \frac{3}{5}\right)$

$v = x_1 + 1 = \frac{7}{5}$

$E(1, Y^*) = 2y_1 - 1(1-y_1) \leq v$

$E(2, Y^*) = y_1 + 3(1-y_1) \leq v$

intersection $\quad 3y_1 - 1 = -2y_1 + 3$

$5y_1 = 4$

$y_1 = \frac{4}{5} \qquad Y^* = \left(\frac{4}{5} \; \frac{1}{5}\right)$

Problem 8: (16 points) Consider the constant sum game with the following payoff matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

For each of the following pairs of mixed strategies, determine if it is a saddle point, and if not, find player I's best response to the given strategy for player II (find the best response to Y_i).

(a) $X_1 = (\frac{2}{3}\ 0\ \frac{1}{3}), Y_1 = (\frac{2}{3}\ \frac{1}{3}\ 0)$

(b) $X_2 = (0\ \frac{1}{2}\ \frac{1}{2}), Y_2 = (\frac{1}{4}\ \frac{1}{2}\ \frac{1}{4})$

$$E(X_1,Y_1) = \frac{2}{3}\cdot\frac{2}{3}\cdot 1 + \frac{2}{3}\cdot\frac{1}{3}\cdot 2 + \frac{1}{3}\cdot\frac{2}{3}\cdot 2 + \frac{1}{3}\cdot\frac{1}{3}\cdot 0$$

$$= \frac{4}{9} + \frac{4}{9} + \frac{4}{9} = \frac{4}{3}$$

$E(X_1,1) = \frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 2 = \frac{4}{3} \geq \frac{4}{3}$ ✓

$E(X_1,2) = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 0 = \frac{4}{3} \geq \frac{4}{3}$ ✓

$E(X_1,3) = \frac{2}{3}\cdot 3 + \frac{1}{3}\cdot 1 = \frac{7}{3} \geq \frac{4}{3}$ ✓

$E(1,Y_1) = \frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 2 = \frac{4}{3} \leq \frac{4}{3}$ ✓

$E(2,Y_1) = \frac{2}{3}\cdot 0 + \frac{1}{3}\cdot 3 = \frac{3}{3} \leq \frac{4}{3}$ ✓

$E(3,Y_1) = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 0 = \frac{4}{3} \leq \frac{4}{3}$ ✓

(X_1,Y_1) is an equilibrium

$$E(X_2,Y_2) = \frac{1}{2}\cdot\frac{1}{4}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2}\cdot 3 + \frac{1}{2}\cdot\frac{1}{4}\cdot 1 + \frac{1}{2}\cdot\frac{1}{4}\cdot 2 + \frac{1}{2}\cdot\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{4}\cdot 1 = \frac{3}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{5}{4}$$

$$\neq \frac{4}{3}$$

not equilibrium

$E(1,Y_2) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} = 2$ max

$E(2,Y_2) = 0\cdot\frac{1}{4} + 3\cdot\frac{1}{2} + 1\cdot\frac{1}{4} = \frac{7}{4}$

$E(3,Y_2) = 2\cdot\frac{1}{4} + 0 + 1\cdot\frac{1}{4} = \frac{3}{4}$

$(1\ 0\ 0)$ is best response