Problem 4 (15 points): Consider a solitaire version of Shut the Box in which the player starts with tiles numbered $\{1, \ldots, 9\}$ open and on each turn chooses to roll either one or two fair six-sided dice (the player always has the choice of how many dice to roll), and either chooses a subset of open tiles to close whose sum equals the sum of the dice, or starts a new turn if there is no such subset. The goal is to close all the tiles in as few turns as possible.

Write a recurrence (including base case(s)) for $E(S)$, the expected number of turns to finish the game starting with set of open tiles S and assuming optimal play. You may use $P_r(n, t)$ to denote the probability of rolling a total of t with n dice.

$E(\emptyset, t) = 4 - t$

$E(S, 4) = 0$

$$E(S,t) = \max_{d=1,2} \sum_{t=d}^{6d} P_r(t,d) \cdot \begin{cases} E(S, t+1) & \text{if no } S' \subseteq S \text{ has sum}(S') = t \\ \max_{\substack{S' \subseteq S \\ \text{sum}(S') = t}} E(S - S', t)) \end{cases}$$

Problem 5 (15 points): Consider a variation of the solitaire version of Shut the Box in which the tiles reset to what they were at the start of the turn if the player rolls and cannot shut any tiles (because there is no subset of open tiles whose total equals the sum on the dice). The player may choose to start a new turn before any roll. For example, starting with $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, if the player rolls a 1+1 and closes the 2, and then rolls another 1+1, then the first turn ends after those two rolls and the game resets to $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ because there is no set of tiles that can be closed. But if the player had started a new turn after the first roll of 1+1 then the second turn ends after the second roll of 1+1 but the open tiles would be $\{1, 3, 4, 5, 6, 7, 8, 9\}$ because those were the open tiles at the start of the second turn.

Let $E(S, T)$ denote the expected number of turns required to finish the game given that the set of tiles open at the start of the current turn was S and the current set of tiles open is T. Write a recurrence (including base case(s)) for $E(S, T)$, and describe very briefly the techniques you would use to compute the table of $E(S, T)$ values.

$E(\emptyset, T) = 1$

$$E(S,T) = \min\left(1 + E(T,T),\ \min_{d=1,2} \sum_{t=d}^{6d} P_r(t,d) \cdot \begin{cases} 1 + E(S,S) & \text{if no } T' \subseteq T \text{ has sum}(T') = t \\ \min_{\substack{T' \subseteq T \\ \text{sum}(T') = t}} E(S, T') & \text{otherwise} \end{cases} \right)$$

dynamic programming + value iteration