Using Sprague-Grundy

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

.	**K0**	**{}**	***0**
x	**K1**	**{.} = {*0}**	***1**
xx	**K2**	**{.., x., .x} = {*0, *1, *1}**	***2**
xxx	**K3**	**{.xx, x.x, ..x} = {*2, (*1 xor*1) = *0, *1}**	***3**
xxxx	**K4**	**{.xxx, x.xx, ..xx, x..x} = {*3, *(1 xor 2) = *3, *2, *(1 xor 1) = *0}**	***1**
xxxxx	**K5**	**{.xxxx, x.xxx, xx.xx, ..xxx, x..xx} = {*1, *(1 xor 3) = *2, *(2 xor 2)=*0, *3, *(1 xor 2)=*3}**	***4**

Properties of Equivalence

For all finite, impartial, normal games G, H, K

$G \approx H \rightarrow$

$G \approx$

$G \approx H$

$G \approx H$ and $H \approx K$

$G + H \approx$

$(G + H) + K \approx$

Lemmas

L1: Any position $G+H$ is an N position if G, H are in different outcome classes and is a P position if G, H are both P positions.

Proof: (Induction on length of game: $\forall n \geq 0$, $\forall$ pos $G+H$ of length n, ...)

Base case: $(n=0)$ Then $G+H = \{\}$ so $G=$ $H=$

Ind step: Suppose $G+H$ has length $k>0$ and suppose sums of length $<k$ satisfy

3 cases 1) G in N, H is P

2) G is P, H is N

3) G, H both P

So every G' is N and every H' is N

Ind hyp applies to each $G'+H$ or $G+H'$: all are $N+P$ or $P+N$, so all are N

So $G+H$ is P

L2: For every P position A and every position G, $G+A \simeq G$

Proof: Suppose A is a P position and G is any position

Let H be any pos.

Two cases: $G+H$ is P $\quad G+A+H \simeq G+H+A$

$G+H$ is N $\quad G+A+H \simeq \underbrace{G+H}_{N}+\underbrace{A}_{P}$

N pos (L1)

L3: $G \approx G'$ if and only if $G+G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$. Then $G+G$, $G+G'$ have same outcome class
$G+G$ is P
so $G+G'$ is P too

$\leftarrow$: Suppose $G+G'$ is a P position

Then $G+(G+G') \approx G$ (L2)

and $G'+(G+G) \approx G'$ (L2)

associative
commutative

so $G \approx G+(G+G') \approx (G+G)+G' \approx G'+(G+G) \approx G'$

so $G \approx G'$ (transitive)

L4: If $G = \{G_1, \ldots, G_\ell\}$ and $G_1 \approx *n_1$ and $\cdots$ and $G_\ell \approx *n_\ell$
then $G \approx \{*n_1, \ldots, *n_\ell\}$

[use L3: show $G + \{*n_1, \ldots, *n_\ell\}$ is P pos]

Consider options of $G+\{*n_1, \ldots, *n_\ell\}$

1) $\underbrace{G_i + \{*n_1, \ldots, *n_\ell\}}$, (move on G to one of $G_1, \ldots, G_\ell$)

N pos b/c has option $G_i + *n_i$ where $G_i \approx *n_i$, which is a P pos (L3)

2) $\underbrace{G + *n_i}$, (move on $\{*n_1, \ldots, *n_\ell\}$)

N pos b/c has option $G_i + *n_i$, which is a P pos

All options are N pos, so $G + \{*n_1, \ldots, *n_\ell\}$ is a P pos

so $G \approx \{*n_1, \ldots, *n_\ell\}$ (L3)

Sprague-Grundy

Every finite, impartial normal game is equivalent to some nimber.

Proof :

Base case $(n=0)$: only game with length 0 is $\{\ \} = *0 \approx *0$

Induction step: Let G be a game of length $k>0$ and suppose all games G' of length $<k$ are equivalent to some nimber.

So by induction hypothesis,

Claim: $G' + *m$ is P-pos where $m = \operatorname{mex}(\{n_1, \dots, n_\ell\})$
so $G' \approx *m$

Consider all options of $G' + *m$
Three cases: i) $G' + *j$,

ii) $*i + *m$

iii) $*i + *m$,

iv) $*i + *m$,

All options of $G' + *m$ are N-positions
So $G' + *m$ is P

All options of $G' + *m$ are N-positions

so $G' + *m$ is P

$\therefore$ $G' \approx *m$ (L3)

$G' \approx G$

so $G \approx *m$ (trans)

Nim-sum

Theorem: $*n + *m \approx *(n \oplus m)$

Proof: (induction on length of game, $n+m$)

Base case ($n+m=0$): Then $n=0$, $m=0$, $n \oplus m = 0$

$*n + *m = *0 + *0 = \{\} = *0$

Induction Step: Suppose $n+m>0$ and all n', m' s.t. $n'+m' \le n+m$ have $*n' + *m' \approx *(n' \oplus m')$

$$*n + *m = \{*0 + *m, \dots, *(n-1) + *m, *n + *0, \dots, *n + *(m-1)\}$$

$$\approx \{*(0 \oplus m), \dots, *((n-1) \oplus m), *(n \oplus 0), \dots, *(n \oplus (m-1))\}$$

$$\approx *\mathrm{mex}(\{0 \oplus m, \dots, (n-1) \oplus m, n \oplus 0, \dots, n \oplus (m-1)\})$$

Claim: $\mathrm{mex}(\{0 \oplus m, \dots, (n-1) \oplus m, n \oplus 0, \dots, n \oplus (m-1)\}) = n \oplus m$

1) $n \oplus m$ is excluded: suppose $n \oplus m = i \oplus m$, $i<n$

then $n \oplus m \oplus m = i \oplus m \oplus m$

$n = i$

suppose $n \oplus m = n \oplus i$, $i<m$

then $n \oplus n \oplus m = n \oplus n \oplus i$

$m = i$

2) All x s.t. $0 \le x < n \oplus m$ are included:

Find most significant bit where x, $n \oplus m$ differ

That bit is in $n \oplus m$ and in x

To be in $n \oplus m$, corresponding bits in n, m are

Assume, WLOG, bits are in n, in m

So $m \oplus x < n$

and $*(m \oplus x) + *m$ is an option of $*n + *m$

But $*(m \oplus x) + *m \approx *(m \oplus x \oplus m)$

$= *x$

Combinatorial Games

Nim, Kayles | Chess, Checkers, Go | Backgammon, Yahtzee | Poker | Roshambo | Starcraft

Combinatorial Game:

two-player

turn-based

non-stochastic

perfect information

impartial

normal

misere

finite

Divisors: Start with 1...n, players take turns taking a number with remaining divisors; opponent gets all the remaining divisors. Game is over when no moves remain; winner is player with higher sum (draw if =)

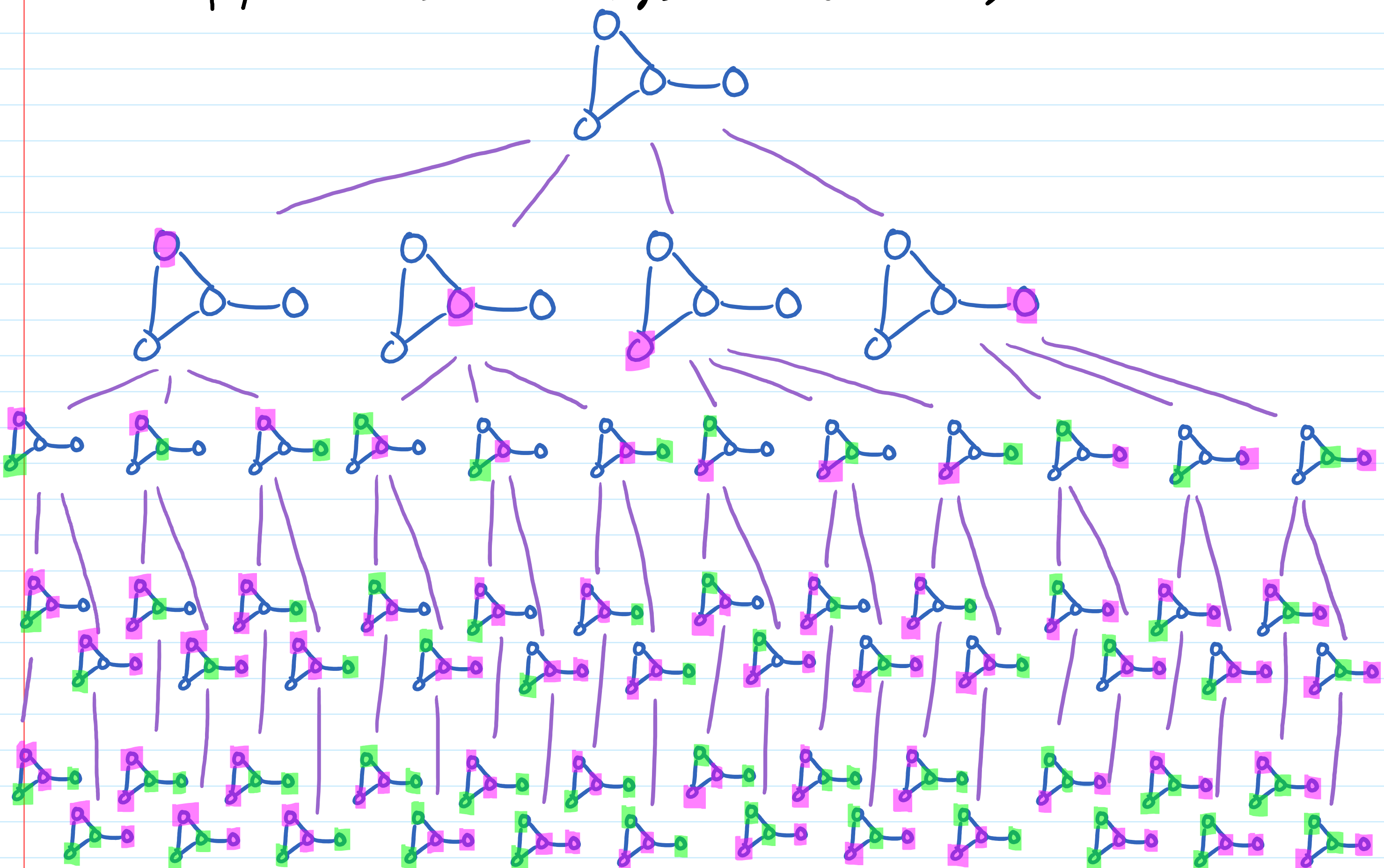
P1 1 2 3 4 5 6

P2 2|1|3456 3|1|2456 4|12|356 5|1|2346 6|123|45

P1 23|16|45 32|14|56 32|16|45 43|126|5 52|41|36 523|16|4

526|413|

Graph: take turns coloring a vertex in a graph with your color player who covers the most edges wins (draw if =)

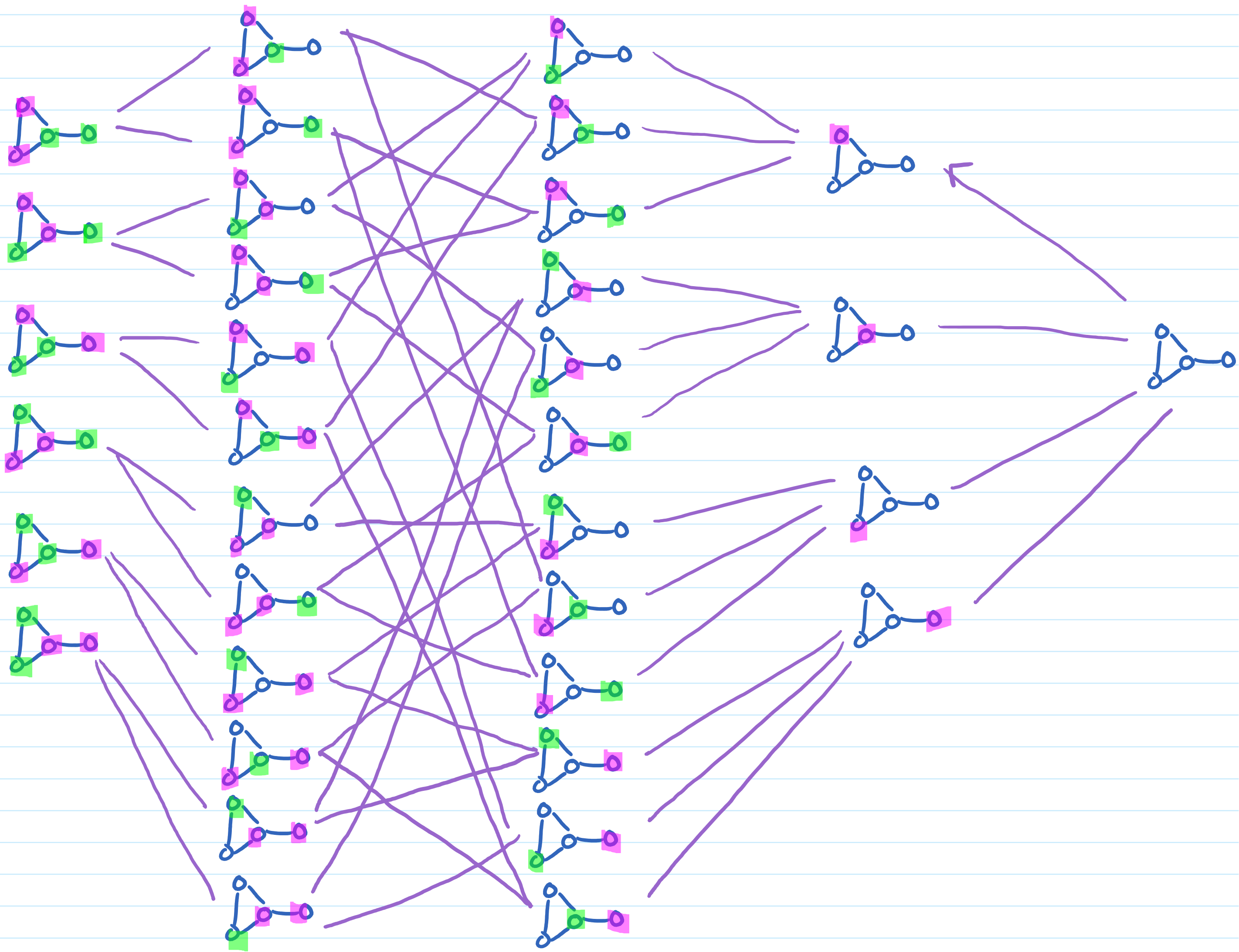


Dynamic Programming

Order positions by maximum distance to end.

Determine winner of distance 0 positions (end) by

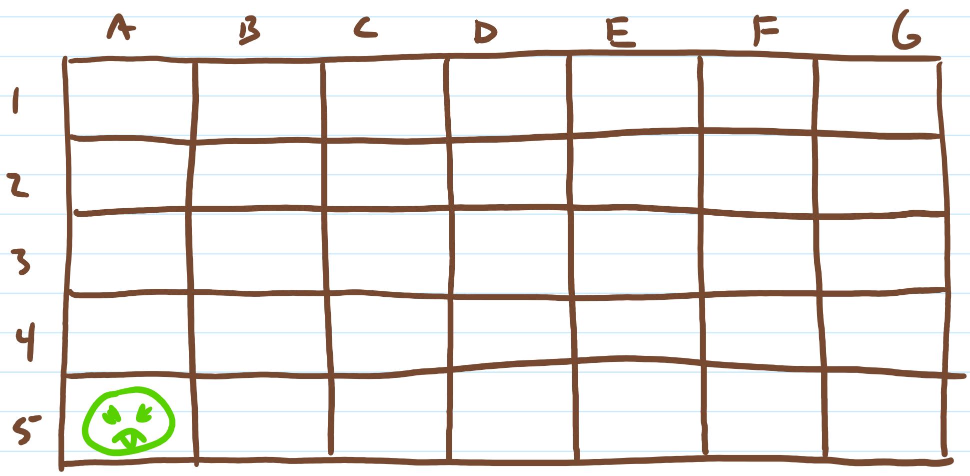
Use recursive formula to determine value of other positions in order of



Chomp

Play on m × n grid. Take turns selecting a remaining cell, remove all above and to right.

Last move loses



Outcome class = who has winning strategy

N next player wins
P prev player wins

Any position in a finite, impartial, normal or misere game is

000
100
110
111
200
210
211
220
221
222