Combinatorial Games

	Nim, Kayles	Chess, Checkers, Go	Backgammon, Yahtzee	Poker	rock-paper scissors Roshambo	Starcraft
Combinatorial Game:						
✓ two-player	✓	✓		> 2		
✓ turn-based	✓	✓			X	X
✓ non-stochastic no element of chance	✓	✓	X	X		X
✓ perfect information you know everything – all possible moves	✓	✓		X		X
normal – last move wins	✓					
misere – last move loses						
finite – bound on total # of moves	✓					

https://xkcd.com/1002/

	Nim, Kayles	Chess, Checkers, Go
– impartial no ownership of pieces moves don't depend on turn	✓	X

Backtracking

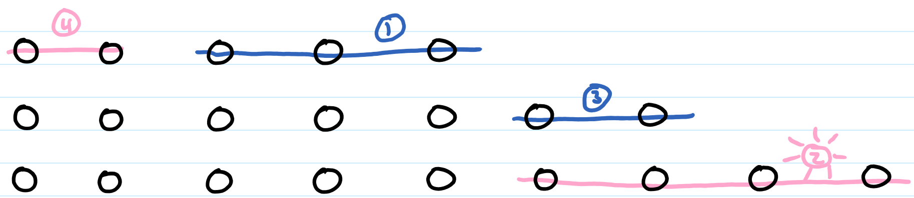
```
find_solution(s)
  if s is solved position return []

  for every possible move m
    update s to reflect move m
    solution =  find_solution(s)
    if solution is not NIL
      return [m] + solution
    else
      undo move m in s
  return NIL
```

→ make a copy of s that reflects move m

(don't need undo if made a copy)

Nim



Start with rows of $n_1, n_2, \ldots, n_k$ stones

On each turn, take as many stones as you wish from one row

If no possible moves, you lose (last move wins)

0010
0111
1001   0101
1100

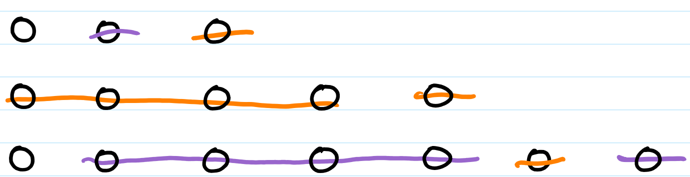
0010
0101
0101
10

DEF: If player to make next move has a winning strategy, position is an N position; otherwise a P position

game is over and you won
or
there is a move [so that for all opp moves you still have winning strategy] opponent goes to P position

THM: Nim position is P if and only if xor is 0

COR: winning move makes xor 0



011
xor 101
xor 111
001 → xor
leave → 110 = 6

1) compute xor of numbers of stones, x
2) find most sig 1 in result
3) find row r with that bit set
4) remain ← x xor count(r)
5) take count(r) − remain from r

Proof: Induction on number of stones (for all n, positions with n stones obey rule)

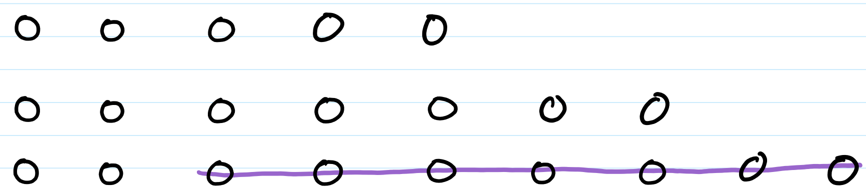
Base case: n = 0 → already lost so P

Induction: Let n > 0 and assume positions with m < n stones are P if and only if xor is 0

If xor is 0 — all moves make it non-zero, ind. hyp applies —

If xor is 0 — all moves make it non-zero
ind. hyp applies
all moves are to N so pos is P

If xor is non-zero — 1-5 above make it 0
ind hyp applies
result is P
start is N



0101
xor 0111
xor 1001
1011

101
111
010
---
000 P

1001
xor 1011

0010 leave 2

1-2-3 Nim

Start with one rows of n stones

On each turn, take as 1, 2, or 3 stones one row

If no possible moves, you lose (last move wins)

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	N	N	N	P	N	N	N	P	N	N	N	P	N

Prove (ind) that n is P $\iff n \equiv 0 \pmod 4$
(n is a mult of 4)

Start with three rows of $n_1, n_2, \ldots, n_k$ stones

On each turn, take as 1, 2, or 3 stones one row

If no possible moves, you lose (last move wins)