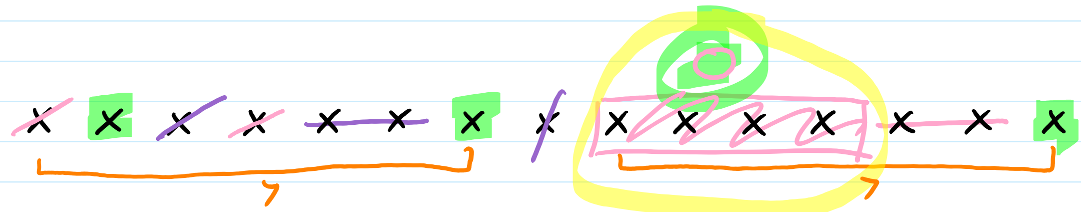
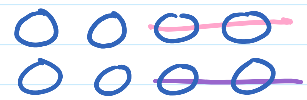
Kayles

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7



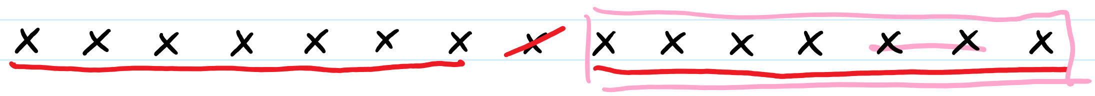
Start with row of n pins

On each turn, take 1 or 2 adjacent pins



If no possible moves, you lose

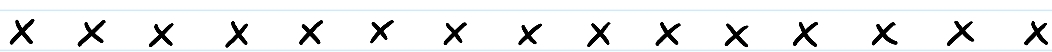
normal - last move wins



K15

K7 + K7

K7 + K4 + K1



Game Positions

Game position = set of positions you can move to

In traditional 1-row Nim

$$\sqcup = \{\} = \emptyset$$

$$0 = \{\sqcup\}$$

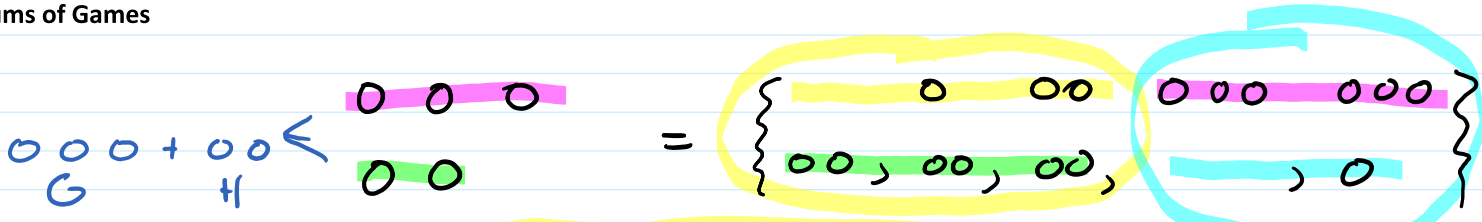
$$00 = \{\sqcup, 0\}$$

$$000 = \{\sqcup, 0, 00\}$$

$$0000 = \{\sqcup, 0, 00, 000\}$$

$$\vdots$$

Sums of Games



$$\underline{G + H} = \{ G' + H \mid G' \text{ is an option of } G \} \cup \{ G + H' \mid H' \text{ is an option of } H \}$$

Equivalence of Games

XXXX $\approx$ O

For impartial, normal games G, G', say $G \approx G'$ if and only if for any game H, $G + H$ $G' + H$ are both N pos (winning) or both P pos (losing)

outcome class

oo o

Is $*2 \approx *1$? Nimbers

NO

$*2 + \underline{*2}$ (G, H): oo / oo P

$*1 + \underline{*2}$ (G', H): o / o~~o~~ N

P K4 + K1 + K4 + K1

N K3 + K1 + K4 + K1

N (K3 + K1 + K1) + (K4) G

N (K3 + K1 + K1) + (0) G'

H

Is $*5 \approx *3$? NO

$*5 + \underline{*3}$: ooo~~oo~~ / ooo N

$*3 + \underline{*3}$: P

Conjecture: $\forall m, n \in \mathbb{N}$, $m \neq n \rightarrow *m \not\approx *n$

$*m \approx *n \Updownarrow n = m$

Is $*2 + *1 \approx *3$ YES

oo / o ooo

$*2 + *1 + \underline{*0}$: N $*3 + \underline{*0}$: N

$*2 + *1 + \underline{*1}$: N $*3 + \underline{*1}$: N

$*2 + *1 + \underline{*2}$: N $*3 + \underline{*2}$: N

$*2 + *1 + \underline{*3}$: oo / o / ooo P $*3 + \underline{*3}$: ooo / ooo P

$*2 + *1 + \underline{*4}$: oo / o / ooo~~o~~ N $*3 + \underline{*4}$: ooo / ooo~~o~~ N

Conjecture: $*n + *m \approx *(n \oplus m)$

bitwise exclusive or

Properties of Equivalence

For all finite, impartial, normal games G, H, K

$G \approx H \longrightarrow G, H$ have same outcome class

$\underset{G}{\overset{\shortparallel}{G + \approx 0}}$ $\underset{H}{\overset{\shortparallel}{H + \approx 0}}$

equivalence relation

reflexive $G \approx G$

symmetric $G \approx H \longrightarrow H \approx G$

transitive $\underline{G \approx H}$ and $\underline{H \approx K} \longrightarrow G \approx K$

$G + H \approx H + G$ commutative

$(G+H)+K \approx G+(H+K)$ associative

Lemmas

L1: Any position $G+H$ is an N position if G, H are in different outcome classes and is a P position if G, H are both P positions.

$N+P=N$
$P+N=N$
$P+P=P$
$N+N=??$

$\underset{N}{*3}+\underset{N}{*2}$ is N

$\boxed{*3+*3 \text{ is } P}$

Proof: by induction on length of game

$*0$ is P

P positions act as 0 in +

L2: For every P position A and every position G, $G+A \approx G$

Proof: Suppose A is a P position and G is any position

L3: $G \approx G'$ if and only if $G+G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$.

$\leftarrow$: Suppose $G+G'$ is a P position

THM (Sprague-Grundy) Every finite, impartial normal game is equivalent to some nimber.

Proof: by induction on length of game

COR: Let $G = \{G_1, \ldots, G_m\}$ where $G_1 \approx *n_1$, $\ldots$ $G_m \approx *n_m$

then $G \approx *\,\mathrm{mex}(n_1, \ldots, n_m)$ $\quad \mathrm{mex}(\{0,1,4,7,8\}) = 2$

mex ↳ minimum excludant = smallest nonneg int not in set

COR: If $G_1 \approx *n_1$ and $G_2 \approx *n_2$ then $G_1 + G_2 \approx *(n \oplus m)$

Proof:

Kayles

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

$\sqcup = \{\ \}$ $\quad \text{mex}(\emptyset) = 0$ $\quad *0$

$x = \{\sqcup\} = *\text{mex}(\{0\})$ $\quad *1$

$x \quad x = \{x\} \approx \{*1\}$ $\quad \text{mex}(\{1\})\ *0$

$xx = \{x, \sqcup\}$ $\quad \text{mex}(\{0,1\}) = 2$ $\quad *2$

$\quad\ \ *1 \quad *0$

$xxx = \{xx, x\ \ x, x\}$ $\quad \text{mex}(\{0,1,2\})$ $\quad *3$

$\quad *2 \quad x+x \quad *1$

$\quad *1+*1$

$\quad *0$

$x \quad x = x + x$

$\approx *1 + *1$

$\approx *(1 \oplus 1)$

$\approx *0$

$xxxx = \{xxx, x\ \ xx, xx, x\ \ x\}$ $\quad \text{mex}(\{3,2,0\})\ *1$

$*3 \quad *1 \oplus *2 \quad *2 \quad *1 \oplus *1$

$*3 \quad *0$

$xxxxx = \{xxxx, xxx\ \ x, xx\ \ xx, xxx, x\ \ xx\}$ $\quad *4$

$*1 \quad *(3 \oplus 1) \quad *(2 \oplus 2) \quad *3 \quad *(1 \oplus 2)$

$*2 \quad *0 \quad *3$

$$\begin{array}{r} 01 \\ 10 \\ \hline 11 \end{array}$$

$xxxxxx =$

9	6	1	7
~~xxxxxxxxx~~	xxxxxx	x	xxxxxxx
$*4$	$*3$	$*1$	$*2$

$*3 + *3$

$*(3 \oplus 3)$

$*0$

$$\begin{array}{r} 100 \\ 011 \\ 001 \\ 010 \\ \hline 100 \end{array}$$

$$\begin{array}{r} 100 \\ 100 \\ \hline 000 \end{array}$$

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