Simultaneous Play Games

		II: R	P	S
I	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

zero-sum = I's reward + II's reward = 0

payoff matrix for I

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

II's payoffs

$$B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

a_{ij} = payoff for I if I plays i + II plays j

zero-sum $a_{ij} + b_{ij} = 0 \;\forall i,j$

constant-sum $a_{ij} + b_{ij} = C \;\forall i,j$

constant-sum game A w/ constant C

no strategy
1) play zero-sum game on $A - C$
2) award player II C points

Penalty Kick

II goalkeeper

		L	R
I - taking shot	L	$\frac{1}{8}, -\frac{1}{8}$	$\frac{1}{4}, -\frac{1}{4}$
	R	$\frac{1}{4}, -\frac{1}{4}$	$\frac{1}{6}, -\frac{1}{6}$

$$A = \begin{pmatrix} \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{6} \end{pmatrix}$$

II

		W	X	Y	Z
I	A	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
	B	-1	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$
	C	$\frac{1}{2}$	0	-1	$\frac{1}{2}$

Stag Hunt

		S	H
I	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

non-constant sum

$R > T = P > S$

Prisoners Dilemma

(say nothing) → Coop

	Coop	Defect
Coop	R 3, 3 R	S 0, 5 T
Defect	T 5, 0 S	P 1, 1 P

Defect (implicate partner)

$T > R > P > S$

reward = years off prison sentence

R = reward for cooperating
P = "punishment" for defecting
S = reward for being a sucker (coop while partner defects)
T = reward for giving in to temptation (defect while partner coop)

	R	P	S	
Rock	0	-1	1	-1
Paper	1	0	-1	-1
Scissors	-1	1	0	-1
	$v^+ = 1$ 1	1	1	$v^- = -1$

→

	L	R	
L	$\frac{1}{2}$	1	$\frac{1}{2}$
R	1	$\frac{2}{3}$	$\frac{2}{3}$
	$v^+ = 1$ 1	1	$v^- = \frac{2}{3}$

equilibrium

	W	X	Y	Z	
A	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$
B	-1	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$	-1
C	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	-1
	$v^+ = \frac{1}{2}$ $\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$v^- = \frac{1}{2}$

v^- amt I guaranteed to win $= \max_i \min_j a_{ij}$

$\min_j a_{ij}$ = min of row i

v^+ = ceiling on what II gives up $= \min_j \max_i a_{ij}$

For any constant-sum game, $v^- \le v^+$

$\forall i,j \quad \min_{j'} a_{ij'} \le a_{ij}$ (def min)

$\forall j \quad \max_i \min_{j'} a_{ij'} \le \max_i a_{ij}$

$\cancel{\min_j} \max_i \min_{j'} a_{ij'} \le \min_j \max_i a_{ij}$

$\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}$

$v^- \le v^+$

Saddle point: where neither player has incentive to unilaterally change strategy
(equilibrium) → pair of strategies where if 1 player changes and other doesn't, reward for ...

changes is no better

i^*, j^* is saddle point $a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$

A constant-sum game A has a saddle point in pure strategies if and only if $v^- = v^+$

↳ always pick one row or column

$\Rightarrow$: Suppose A has a saddle point in pure strategies.

Then $\exists\ i^*, j^*$ s.t. $a_{ij^*} \le a_{i^*j^*}$ for all i $\quad a_{i^*j^*} \le a_{i^*j}$ for all j

so $\max_i a_{ij^*} \le a_{i^*j^*}$ for all i $\quad a_{i^*j^*} \le \min_j a_{i^*j}$ for all j

bigger than all in max $\quad$ smaller than all in min

and $\min_j \max_i a_{ij} \le \underbrace{\max_i a_{ij^*}}_{\text{term is in min}}$ $\quad \underbrace{\min_j a_{i^*j}}_{\text{term is in max}} \le \max_i \min_j a_{ij}$

so $v^+ \le \max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j} \le v^-$

but $v^- \le v^+$ so all $\le$ are $=$ (squeeze)

$\Leftarrow$ Suppose $v^- = v^+$. Let i^* be the i s.t. $v^- = \max_i \min_j a_{ij}$

j^* be the j s.t. $v^+ = \min_j \max_i a_{ij}$

$a_{i^*j} \ge \min_j a_{i^*j} = v^+ = v^- = \max_i a_{ij^*} \ge a_{ij^*}$

in min $\quad$ in max

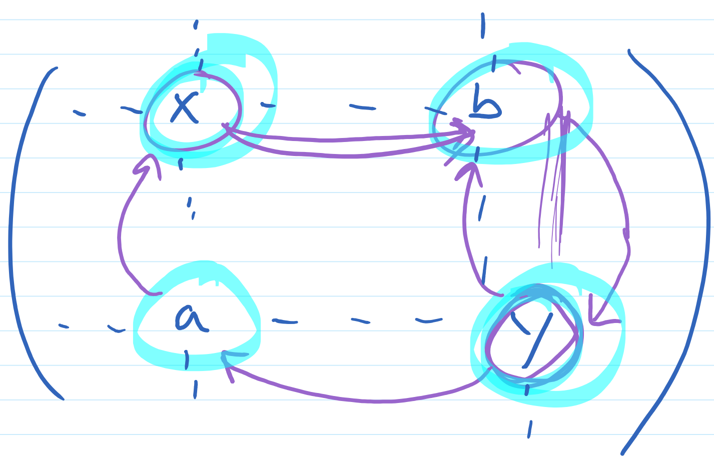
$a_{i^*j^*} \ge v^+ = v^- \ge a_{i^*j^*}$

and $\ge$ are $=$ $\quad$ squeeze

$\therefore\ v^+ = v^- = a_{i^*j^*}$

$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$ $\quad$ for all i, j

Suppose there are 2 saddle points in pure strategies (i_1, j_1) and (i_2, j_2)
with values $a_{i_1j_1} = v_1$ $\quad a_{i_2j_2} = v_2$



$x \le b \le y \le a \le x$

$= \quad = \quad = \quad =$

Mixed Strategies - probability distribution over rows (I) or columns (II)

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$X = (x_1 \ldots x_n)$ $\quad 0 \le x_i, y_i \le 1$

$Y = (y_1 \ldots y_m)$ $\quad \sum x_i = 1 = \sum y_i$

I chooses i w/ prob x_i

II chooses j w/ prob y_j

$$X^* = \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right) = Y^*$$

$$\underset{\text{expected reward for I}}{E}(\underset{\sim}{X}, \underset{\sim}{Y}) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I plays } i, \text{ II plays } j)\cdot a_{ij}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I plays } i) P(\text{II plays } j)\cdot a_{ij}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\cdot x_i \cdot y_j$$

$$= X A Y^T$$

$\tfrac{1}{3}\cdot 0 + \tfrac{1}{3}\cdot 1 + \tfrac{1}{3}\cdot -1$

$$\underset{1\times 3}{\begin{pmatrix} \tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} \end{pmatrix}} \underset{3\times 3}{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}} \begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ \tfrac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ \tfrac{1}{3} \end{pmatrix} = 0$$

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

X^*, Y^* is a saddle point in mixed strategies if and only if

II changes - worse for II (better for I)

$$E(X, Y^*) \underset{=}{\le} E(X^*, Y^*) \underset{=}{\le} E(X^*, Y) \quad \text{for all } X, Y$$

I changed → worse for I

Every game has a saddle point in mixed strategies

Equilibrium Theorem: If X^*, Y^* is a saddle point in mixed strategies for A with $x_i > 0$ $y_j > 0$ then

$$E(X^*, j) = E(i, Y^*) = \text{value}(A)$$

Best Response: Best response to a mixed strategy X is Y that maximizes XAY^T

Strategies in saddle point are best responses to each other

Finding Saddle Points in Mixed Strategies

Thm: X^*, Y^* is a saddle point in mixed strategies and value$(A) = E(X^*, Y^*)$ if and only if

$$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j) \quad \text{for all } i, j$$

pure strategy: col j all the time

$Y = (0\ 0 \cdots 0 \cdots 1 \cdots\cdots 0)$

column j

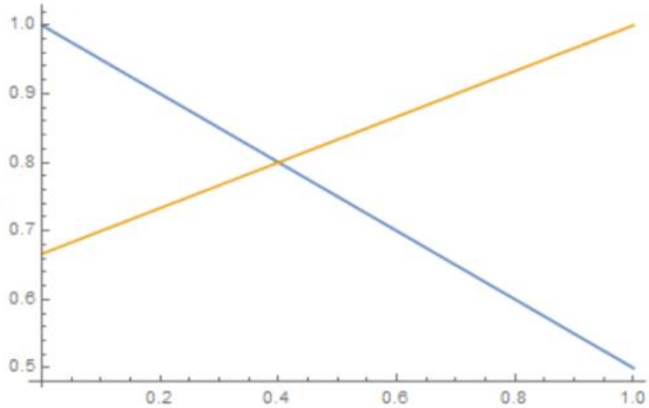
	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Suppose there are two saddle points in mixed strategies (X_1^*, Y_1^*) (X_2^*, Y_2^*)

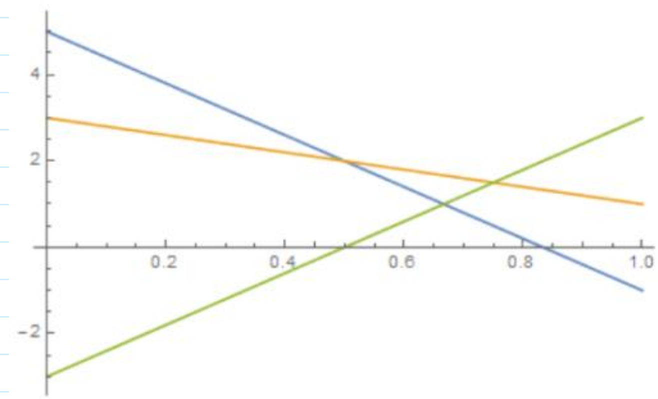
$$\begin{aligned} E(X_1^*, Y_2^*) &\le E(X_2^*, Y_2^*) \\ &\le E(X_2^*, Y_1^*) \\ &\le E(X_1^*, Y_1^*) \\ &\le E(X_1^*, Y_2^*) \end{aligned}$$

Can use to find solutions

	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$



$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$



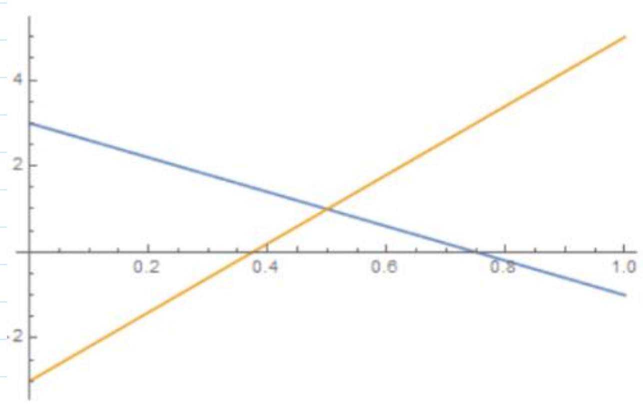
$E(x,1) =$

$E(x,2) =$

$E(x,3) =$

$E(1,y) =$

$E(2,y) =$



$$A = \begin{pmatrix} 0 & -1 & 3 \\ 2 & 5 & -3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$